Optimized robustly controlled invariant sets for constrained linear discrete-time systems

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Abstract

In this paper we introduce the concept of optimized robust controlled invariance for a discrete-time, linear, time-invariant system subject to additive state disturbances. It is assumed that the disturbance is bounded, persistent and acts additively on the state. Novel procedures for the computation of robustly controlled invariant sets and corresponding controllers are presented. These results are useful in robust optimal control of constrained discrete-time, linear, time-invariant system subject to additive state disturbances. Their application to robust optimal control and robust model predictive control is illustrated.

Keywords: Set invariance, robust control, linear systems, robust time minimal control, robust model predictive control.

1 Introduction

The theory of set invariance plays a fundamental role in the control of constrained systems and has been a subject of research by many authors — see for instance [Aub91, Bla99, Ker00] and the references therein. An interesting study of the set invariance in non-cooperative games is initiated and elaborated in [CD00, Car00].

Two important issues are the calculation of the minimal robustly positively invariant (mRPI) set and the maximal robustly positively invariant (MRPI) set.

The mRPI set is used as a target set in robust time-optimal control [MS97], in , directly or indirectly, the design of robust predictive controllers [ML01, LCRM04, KM03a, KMss, Ker00, CRZ01] and in understanding the properties of the *maximal* robustly positively invariant set [KG98, Kou02].

Despite this wide use of the mRPI and MRPI sets, there are still unresolved issues. For the case of the mRPI set, there exists no method for the exact computation of the mRPI set, except those given in [Las93, Sect. 3.3], [MS97, Thm. 3] and [SM98, Sect. II.B], where it is assumed that the closed-loop system dynamics are nilpotent.

In [Kou02, RKKM03] this assumption is relaxed and a method for computing a robustly positively invariant approximation of the mRPI set is investigated and a solution is obtained for a specific case. These results are generalized in [RKKM04]. It is the purpose of this paper to provide methods for computation of polytopic robustly controlled invariant sets via optimization and to demonstrate their use in efficient robust model predictive control.

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This paper is organized as follows. Section 2 is concerned with the necessary definitions and the problem formulation. Section 3 addresses the robust control invariance issue. Section 4 deals with applications of the results to robust optimal control. In Section 5 we illustrate application of our results to robust model predictive control. A few illustrative examples are provided in Section 6. Finally, Section 7 presents conclusions.

2 Preliminary Definitions and Existing Results

Previous research considered the autonomous discrete-time, linear, time-invariant (DLTI) system:

$$x^+ = A_c x + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, x^+ is the successor state and $w \in \mathbb{R}^n$ is an unknown disturbance and $A_c \in \mathbb{R}^{n \times n}$ is a strictly stable matrix (all the eigenvalues of A_c are strictly inside the unit disk). The disturbance w is persistent, but contained in a convex and compact (i.e. closed and bounded) set $W \subset \mathbb{R}^n$, which contains the origin.

The motivation for this paper is that often one would like to determine whether the state trajectory of the system will be contained in a set $X \subset \mathbb{R}^n$, given any allowable disturbance sequence. For this purpose, we present the following definition:

Definition 1 (RPI set). [Bla99] The set $\Omega \subset \mathbb{R}^n$ is a *robustly positively invariant* (RPI) set of (2.1) if $A_c x + w \in \Omega$ for all $x \in \Omega$ and all $w \in W$.

Remark 1. It is useful to note that, by definition, Ω is RPI if and only if $A_c\Omega \oplus W \subseteq \Omega$. Note also that Ω is RPI if and only if $A_c\Omega \subseteq \Omega \ominus W$.

Definition 2 (Constraint-admissible set). The set $\Omega \subset \mathbb{R}^n$ is a *constraint-admissible* set if it is contained in $X \subset \mathbb{R}^n$.

Remark 2. Clearly, the set Ω is a *constraint-admissible*, RPI set if it is contained in X and Ω is RPI.

An important set in the analysis and synthesis of controllers for constrained systems is the minimal RPI set:

Definition 3 (mRPI set). The *minimal robustly positively invariant* (mRPI) set D_∞ of (2.1) is the RPI set of (2.1) that is contained in every closed, RPI set of (2.1).

The properties of the mRPI set D_∞ are well-known. It is possible to show [KG98, Sect. IV] that the mRPI set D_∞ exists, is unique, compact and contains the origin. Moreover, D_∞ is a limit of the sequence $\{D_i\}$ where:

$$D_i \triangleq A_c D_{i-1} \oplus W,\ i \geq 1 \text{ and } D_0 \triangleq \{0\} \tag{2.2}$$

where $\oplus$ denotes the standard Minkowski set addition. It is shown in [KG98, Sect. IV] that $\{D_i\}$ is a Cauchy sequence and that

$$D_\infty = \lim_{i \to \infty} D_i \tag{2.3}$$

where the limit is taken in the Hausdorff metric defined as follows:

Definition 4 (Hausdorff metric). If Ω and Φ are two non-empty, compact sets in $\mathbb{R}^n$, then the *Hausdorff metric* is defined as

$$d_H^p(\Omega, \Phi) \triangleq \max \left\{ \sup_{\omega \in \Phi} d(\omega, \Omega),\ \sup_{\phi \in \Omega} d(\phi, \Phi) \right\}, \tag{2.4}$$

where

$$d(z, \mathcal{Z}) \triangleq \inf_{y \in \mathcal{Z}} \|z - y\|_p. \tag{2.5}$$

Remark 3. Clearly, $\Omega = \Phi$ if and only if $d_H^p(\Omega, \Phi) = 0$. It is also useful to note that $d_H^p(\Omega, \Phi)$ is the size of the smallest norm-ball that can be added to Ω in order to cover Φ and vice versa, i.e.

$$d_H^p(\Omega, \Phi) = \inf \left\{ \varepsilon \geq 0 \;\middle|\; \Phi \subseteq \Omega \oplus \mathbb{B}_p^n(\varepsilon) \text{ and } \Omega \subseteq \Phi \oplus \mathbb{B}_p^n(\varepsilon) \right\}. \tag{2.6}$$

In [Kou02, RKKM03, RKKM04] a method for computation of an invariant approximation of the minimal robustly positively invariant set is given. It is shown that the set $D_{(\zeta,s)}$ defined by:

$$D_{(\zeta,s)} \triangleq (1-\zeta)^{-1} D_s \tag{2.7}$$

where D_s is defined by (2.2):

$$D_s = \bigoplus_{i=0}^{s-1} A_c^i W \tag{2.8}$$

is an invariant approximation of the minimal robustly positively invariant set D_∞ providing that the couple $(\zeta, s) \in [0,1) \times \{0, 1, 2, \ldots\}$ is such that the following set inclusion holds:

$$A_c^s W \subseteq \zeta W \tag{2.9}$$

It is also shown in [RKKM04] that testing whether the set $D_{(\zeta,s)}$ is constraint admissible can be done before its actual computation.

In this note we consider a more general problem in which we do not consider anymore autonomous system (2.1) but general discrete time, linear, time–invariant system and we provide method for computation of a robustly controlled invariant set, that is contained in a *minimal* p norm ball, and corresponding control law via optimization.

Before proceeding we need to define the robustly controlled invariant set for the system $x^+ = f(x, u, w)$, $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}^n$ under the constraints given by: $(x, u, w) \in X \times U \times W$.

Definition 5 (RCI set). The set $\Omega \subset X$ is a *robustly controlled invariant* (RCI) set of $x^+ = f(x, u, w)$ if for all $x \in \Omega$ there exists a $u \in U$ such that $f(x, u, w) \in \Omega$ and all $w \in W$.

We show that our results are better than existing results since our method is based on optimization procedure. Moreover, we illustrate application of our results to robust optimal control and robust model predictive control.

3 Robust Controlled Invariance Issue

We consider the following discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + Bu + w, \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action x^+ is the successor state and $w \in \mathbb{R}^n$ is an unknown disturbance. The disturbance w is persistent, but contained in a convex and compact (i.e. closed and bounded) set $W \subset \mathbb{R}^p$, which contains the origin. Matrices A and B are of appropriate dimensions and couple (A, B) is assumed to be controllable.

If the initial state is x at time 0 (since the system is time-invariant, the current time can always be taken to be zero), then we denote by $\phi(k; x, \pi, w(\cdot))$ the solution to (3.1) at time instant k, given the control policy $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ where for each i, $\mu_i(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^m$ and the disturbance sequence $\mathbf{w} \triangleq \{w_0, w_1, \ldots, w_{N-1}\}$.

The first issue that we are interested in is computation of a robustly controlled invariant set for (3.1). The system satisfies the following equation:

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{3.2}$$

If $x_0 = 0$ and the control action at each time k is:

$$u_k = M_{k-1}w_0 + M_{k-2}w_1 + \ldots + M_1 w_{k-2} + M_0 w_{k-1} \tag{3.3}$$

(i.e. if $u_0 = 0$, $u_1 = M_0 w_0$, $u_2 = M_1 w_0 + M_0 w_1$, ...), then:

$$\begin{aligned} x_{k+1} = (&A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \ldots + ABM_{k-2} + BM_{k-1})w_0 \\ &+ (A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_1 \\ &+ \ldots + (A^2 + ABM_0 + BM_1)w_{k-2} + (A + BM_0)w_{k-1} + w_k \end{aligned} \tag{3.4}$$

Suppose now that the matrices $M_i,\ i = 0, 1, \ldots k-1,\ k > n$ satisfy:

$$A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \ldots + ABM_{k-2} + BM_{k-1} = 0, \tag{3.5}$$

Such a choice always exists, since (A, B) is controllable and $k > n$. It follows that:

$$\begin{aligned} x_{k+1} =&(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_1 \\ &+ \ldots + (A^2 + ABM_0 + BM_1)w_{k-2} + (A + BM_0)w_{k-1} + w_k \end{aligned} \tag{3.6}$$

Since each $w_i \in W$, $x_{k+1} \in F_k$ if $x_0 = 0$, where:

$$\begin{aligned} F_k \triangleq&(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})W \\ &\oplus \ldots \oplus (A^2 + ABM_0 + BM_1)W \oplus (A + BM_0)W \oplus W \end{aligned} \tag{3.7}$$

where $\oplus$ denotes the standard Minkowski set addition.

We are now ready to establish some of the properties of the set F_k.

Proposition 1. *There exists a control law $u : F_k \mapsto \mathbb{R}^m$ such that $Ax + Bu(x) \oplus W \subseteq F_k,\ \forall x \in F_k$, i.e. the set F_k is robustly controlled invariant for system* (3.1)*.*

Proof. Let x be an arbitrary element of F_k. Since $x \in F_k$ it follows by definition of the set F_k:

$$\begin{aligned} x =&(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_0 \\ &+ \ldots + (A^2 + ABM_0 + BM_1)w_{k-3} + (A + BM_0)w_{k-2} + w_{k-1} \end{aligned} \tag{3.8}$$

for some $w_i \in W, i = 0, 1, \ldots k-1$. The last equation can be rewritten in matrix form as:

$$x = D\mathbf{w}, \tag{3.9}$$

for some matrix D easily constructed from (3.8) and the vectorized disturbance sequence $\{w_0, w_1, \ldots w_{k-1}\}$. Let $\mathbf{W}^k \triangleq W \times W \times \ldots \times W$ and for all $x \in F_k$ let $\mathbf{w}^0(x)$ be the unique solution of the following quadratic program:

$$\mathbb{P}_w(x) : \quad \mathbf{w}^0(x) = \arg\min_{\mathbf{w}} \{|\mathbf{w}|^2 \mid \mathbf{w} \in \mathbf{W}^k,\ D\mathbf{w} = x\}, \tag{3.10}$$

Hence, $\mathbf{w}^0(x) = \{w_0^0(x), w_1^0(x), \ldots w_{k-1}^0(x)\}$ and since $x \in F_k$ it follows that:

$$\begin{aligned} x =&(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_0^0(x) \\ &+ \ldots + (A^2 + ABM_0 + BM_1)w_{k-3}^0(x) + (A + BM_0)w_{k-2}^0(x) + w_{k-1}^0(x) \end{aligned} \tag{3.11}$$

Let the control law $u(\cdot)$ be defined by:

$$u(x) \triangleq M_{k-1}w_0^0(x) + M_{k-2}w_1^0(x) + \ldots + M_1 w_{k-2}^0(x) + M_0 w_{k-1}^0(x) \tag{3.12}$$

where $M_i,\ i = 0, 1, \ldots k-1$ satisfy (3.5). If $x \in F_k$, then:

$$\begin{aligned} x^+ =& Ax + Bu(x) + w \\ =& (A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \ldots + ABM_{k-2} + BM_{k-1})w_0^0(x) \\ &+ (A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_1^0(x) \\ &+ \ldots + (A^2 + ABM_0 + BM_1)w_{k-2}^0(x) + (A + BM_0)w_{k-1}^0(x) + w \end{aligned} \tag{3.13}$$

where each $w_i^0(x) \in W, i = 0, 1, \ldots k-1$ by construction and $w \in W$ is arbitrary. Since $M_i,\ i = 0, 1, \ldots k-1$ satisfy (3.5) it follows that:

$$\begin{aligned} x^+ =& (A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_1^0(x) \\ &+ \ldots + (A^2 + ABM_0 + BM_1)w_{k-2}^0(x) + (A + BM_0)w_{k-1}^0(x) + w \end{aligned} \tag{3.14}$$

so that $x^+ = Ax + Bu(x) + w \in F_k$ for all $w \in W$. It follows that $Ax + Bu(x) \oplus W \subseteq F_k$ for all $x \in F_k$ with $u(x)$ defined by (3.12) and (3.10). □

Remark 4. Proposition 1 states that for any $k > n$ the set F_k defined in (3.7), *finitely determined by* k, is robustly controlled invariant for system (3.1).

Remark 5. Since the control $u(x) = \mathbf{M}_k w^0(x)$ and since (3.10) defines a piecewise affine function $w^0(\cdot)$ of state due to the constraint $\mathbf{w} \in \mathbf{W}^k$, it follows that $u(\cdot)$ is a piecewise affine function of state x because it is a linear map of a piecewise affine function.

Remark 6. The condition (3.5) can be relaxed as we will show in the sequel.

3.1 Optimized Robust Controlled Invariance

Suppose that the disturbance polytope is an affine map of a hypercube:

$$W \triangleq \{w = Ed + f \mid |d|_\infty \leq \eta\} \tag{3.15}$$

where $d \in \mathbb{R}^t$, $E \in \mathbb{R}^{n \times t}$ and $f \in \mathbb{R}^n$. We demonstrate how for a given index $k \geq n$ the set F_k and a corresponding control policy that renders the set F_k, defined via matrices $M_i,\ i = 0, 1, \ldots, k-1$, robustly controlled invariant can be computed in a such way that an appropriate norm of F_k is minimized.

Let $\mathbf{M}_k \triangleq (M_0, M_1, \ldots, M_{k-2}, M_{k-1})$, (i.e. $\mathbf{M}_k$ is a matrix formed from the matrices M_i so that $\mathbf{M}_k = [M_0'\ M_1'\ \ldots\ M_{k-2}'\ M_{k-1}']'$) and $D_k = [A^{k-1}B\ A^{k-2}B\ \ldots\ AB\ B]$. Let $\mathbb{M}_k$ denote the set of all matrices $\mathbf{M}_k$ satisfying condition (3.5):

$$\mathbb{M}_k \triangleq \{\mathbf{M}_k \mid A^k + D_k\mathbf{M}_k = \mathbf{0}\} \tag{3.16}$$

Recall that the set $F_k = F_k(\mathbf{M}_k)$ is defined by (3.7):

$$\begin{aligned} F_k(\mathbf{M}_k) \triangleq& (A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})W \\ &\oplus \ldots \oplus (A^2 + ABM_0 + BM_1)W \oplus (A + BM_0)W \oplus W \end{aligned} \tag{3.17}$$

so that:

$$\begin{aligned} F_k(\mathbf{M}_k) =& (A^{k-1} + [A^{k-2}B\ A^{k-3}B\ \ldots\ AB\ B\ \mathbf{0}]\mathbf{M}_k)W \oplus (A^{k-2} + [A^{k-3}B\ A^{k-4}B\ \ldots\ B\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)W \\ &\oplus (A^2 + [AB\ B\ \ldots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)W \oplus (A + [B\ \mathbf{0}\ \ldots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)W \oplus (I_n + [\mathbf{0}\ \mathbf{0}\ \ldots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)W \end{aligned} \tag{3.18}$$

Let $\mathbb{B}_p(\alpha)$ denote the p norm ball in $\mathbb{R}^n$ of radius α:

$$\mathbb{B}_p(\alpha) = \{x \mid |x|_p \leq \alpha\} \tag{3.19}$$

If $p = 1, \infty$ we have

$$\mathbb{B}_p(\alpha) = \{x \mid |x|_p \leq \alpha\} = \{x \mid Dx \leq \alpha d\} \tag{3.20}$$

where D and d are matrices of appropriate dimensions depending on the choice of the norm.

We are interested in computation of a robustly controlled invariant set $F_k(\mathbf{M}_k)$ contained in a *'minimal'* p norm ball, i.e. we wish to find $F_k^0 = F_k(\mathbf{M}_k^0)$ where:

$$(\mathbf{M}_k^0, \alpha^0) = \arg \min_{\mathbf{M}_k, \alpha} \{\alpha \mid F_k(\mathbf{M}_k) \subseteq \mathbb{B}_p(\alpha)\} \tag{3.21}$$

Our next step is to show that our problem can be posed as a linear programming problem if $p = 1, \infty$ by considering a more general problem:

$$(\mathbf{M}_k^0, \alpha^0) = \arg \min_{\mathbf{M}_k, \alpha} \{\alpha \mid F_k(\mathbf{M}_k) \subseteq P(\alpha)\},\ P(\alpha) = \{x \mid Cx \leq \alpha c\},\ \alpha > 0 \tag{3.22}$$

where $P(1)$ is a compact polytope that contains the origin in its interior and $C \in \mathbb{R}^{q \times n}$ and $c \in \mathbb{R}^q$. Before proceeding we need to establish some preliminary results. First, we recall the following definition:

Definition 6 (Support function). The *support function* of a set $\Pi \subset \mathbb{R}^n$, evaluated at $z \in \mathbb{R}^n$, is defined as

$$h(\Pi, z) \triangleq \sup_{\pi \in \Pi} z^T \pi. \tag{3.23}$$

Our main interest in the support function is the well-known fact that the support function of a set allows one to write equivalent conditions for the set to be a subset of another. In particular:

Proposition 2. *Let Π be a non-empty set in $\mathbb{R}^n$ and the polyhedron*

$$\Psi = \left\{\psi \in \mathbb{R}^n \;\middle|\; f_i^T \psi \leq g_i,\ i \in \mathcal{I}\right\}, \tag{3.24}$$

where $f_i \in \mathbb{R}^n$, $g_i \in \mathbb{R}$ and $\mathcal{I}$ is a finite index set.

(i) $\Pi \subseteq \Psi$ *if and only if* $h(\Pi, f_i) \leq g_i$ *for all* $i \in \mathcal{I}$.

(ii) $\Pi \subseteq \mathrm{int}(\Psi)$ *if and only if* $h(\Pi, f_i) < g_i$ *for all* $i \in \mathcal{I}$.

The following result allows one to compute the support function of a set that is the Minkowski sum of a finite sequence of linear maps of non-empty, compact sets and is reported in [RKKM04]:

Proposition 3. *Let each matrix $L_k \in \mathbb{R}^{n \times m}$ and each Φ_k be a non-empty, compact set in $\mathbb{R}^m$ for all $k \in \{1, \ldots, K\}$. If*

$$\Pi = \bigoplus_{k=1}^{K} L_k \Phi_k, \tag{3.25}$$

then

$$h(\Pi, z) = \sum_{k=1}^{K} \max_{\phi \in \Phi_k} (z^T L_k)\phi. \tag{3.26}$$

Furthermore, if $\Phi_k = \mathbb{B}_\infty(1)$, then

$$\max_{\phi \in \Phi_k} (z^T L_k)\phi = |L_k^T z|_1. \tag{3.27}$$

Let $\mathbf{1}_t$ denote vector of ones of length t. Let $\mathrm{abs}(A)$ denote the matrix whose elements are the absolute values of the corresponding components of the matrix A. It is well known, see for example [KM03b], that:

$$\max_d \{a'd \mid |d|_\infty \leq \eta\} = \eta |a|_1.$$

The following observation is a consequence of Proposition 3 and the results are reported in [KM03b, KM04].

Corollary 1. *Let matrices* $A \in \mathbb{R}^{n\times n}$*,* $C \in \mathbb{R}^{q\times n}$*,* $D \in \mathbb{R}^{n\times p}$ *and* $M \in \mathbb{R}^{p\times n}$ *and let* $w \in W$ *where* $W = \{w = Ed + f \mid |d|_\infty \leq \eta\}$ *and* $E \in \mathbb{R}^{n\times t}$ *and* $f \in \mathbb{R}^n$*. Then*

$$\max_{w\in W} C(A + DM)w = \eta \text{abs}(C(A + DM)E)\mathbf{1}_t + C(A + DM)f \tag{3.28}$$

where the maximization is taken row-wise. Moreover, there exists a matrix $L \in \mathbb{R}^{q\times t}$ *such that:*

$$-L \leq C(A + DM)E \leq L \tag{3.29}$$

where the inequality is element-wise, and the solution of (3.28) *is:*

$$\max_{w\in W} C(A + DM)w = \eta L\mathbf{1}_q + C(A + DM)f \tag{3.30}$$

Let

$$\Omega \triangleq \{(\mathbf{M}_k, \alpha) \mid \mathbf{M}_k \in \mathbb{M}_k,\ F_k(\mathbf{M}_k) \subseteq P(\alpha),\ \alpha > 0\} \tag{3.31}$$

where $F_k(\mathbf{M}_k)$ is defined in (3.17) and $P(\alpha)$ is defined (3.22). Consider the following minimization problem:

$$\mathbb{P}_k: \quad (\mathbf{M}_k^0, \alpha^0) = \arg\min_{\mathbf{M}_k,\alpha} \{\alpha \mid (\mathbf{M}_k, \alpha) \in \Omega\} \tag{3.32}$$

Proposition 4. *The minimization problem* $\mathbb{P}_k$ *defined in* (3.32) *is a linear programming problem.*

Proof. The set inclusion $F_k(\mathbf{M}_k) \subseteq P(\alpha)$ holds true if and only if:

$$Cx \leq \alpha c,\ \forall x \in F_k(\mathbf{M}_k) \tag{3.33}$$

or in terms of support functions

$$\max_{x\in F_k(\mathbf{M}_k)} Cx \leq \alpha c, \tag{3.34}$$

where the maximization is taken row-wise. Now, any arbitrary $x \in F_k(\mathbf{M}_k)$ can be written as:

$$\begin{aligned} x = &(A^{k-1} + [A^{k-2}B\ A^{k-3}B\ \dots\ AB\ B\ \mathbf{0}]\mathbf{M}_k)w_0 + (A^{k-2} + [A^{k-3}B\ A^{k-4}B\ \dots\ B\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)w_1 \\ &+ (A + [B\ \mathbf{0}\ \dots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)w_{k-2} + (I_n + [\mathbf{0}\ \mathbf{0}\ \dots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)w_{k-1} \end{aligned} \tag{3.35}$$

where each $w_i \in W,\ i = 0, 1, 2, \dots, k-1$ it follows, by Proposition 3, that:

$$\begin{aligned} \max_{x\in F_k(\mathbf{M}_k)} Cx = &\max_{w_0\in W} C(A^{k-1} + [A^{k-2}B\ A^{k-3}B\ \dots\ AB\ B\ \mathbf{0}]\mathbf{M}_k)w_0 \\ &+ \max_{w_1\in W} C(A^{k-2} + [A^{k-3}B\ A^{k-4}B\ \dots\ B\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)w_1 \\ &+ \dots \\ &+ \max_{w_{k-2}\in W} C(A + [B\ \mathbf{0}\ \dots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)w_{k-2} \\ &+ \max_{w_{k-1}\in W} C(I_n + [\mathbf{0}\ \mathbf{0}\ \dots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k)w_{k-1} \end{aligned} \tag{3.36}$$

From Corollary 1 it follows that there exist a set of matrices $L_i \in \mathbb{R}^{q\times t},\ i = 0, 1, \dots, k-1$ such that:

$$\max_{x\in F_k(\mathbf{M}_k)} Cx = \sum_{i=0}^{k-1} (\eta L_i \mathbf{1}_t + C(A^i + D_i\mathbf{M}_k)f) \tag{3.37}$$

where each $D_i,\ i = 0, 1, \dots, k-1$, is readily constructed from (3.36) and $\Lambda_k \triangleq \{L_0, L_1, \dots, L_{k-1}\}$ and each L_i satisfies:

$$-L_i \leq C(A^i + D_i\mathbf{M}_k)E \leq L_i,\ i = 0, 1, \dots k-1 \tag{3.38}$$

By the basic properties of the Kronecker product it follows that the set inclusion $F_k(\mathbf{M}_k) \subseteq P(\alpha)$ can be expressed as a set of linear inequalities in $(\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$. Similarly, the condition $\mathbf{M}_k \in \mathbb{M}_k$ is a set of linear equalities in $(\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$. Hence, since the cost function in the minimization problem $\mathbb{P}_k$ is a linear function of $(\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$, it follows that the minimization problem $\mathbb{P}_k$ is a linear programming problem. □

Remark 7. Let $\gamma \triangleq (\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \alpha)$ and let:

$$\Gamma \triangleq \{\gamma \mid \mathbf{M}_k \in \mathbb{M}_k,\ \sum_{i=0}^{k-1}(\eta L_i \mathbf{1}_t + C(A^i + D_i\mathbf{M}_k)f) \le \alpha c,$$
$$-L_i \le C(A^i + D_i\mathbf{M}_k)E \le L_i,\ i = 0, 1, \dots k-1,\ \alpha > 0\} \tag{3.39}$$

The minimization problem $\mathbb{P}_k$ is then readily transformed into a linear programming problem:

$$\mathbb{P}_k : \quad \min_{\gamma}\{\alpha \mid \gamma \in \Gamma\} \tag{3.40}$$

3.2 Optimized Robust Controlled Invariance Under Constraints

Suppose now that the system (3.1) is subject to constraints:

$$x \in X = \{x \mid C_x x \le c_x\},\ u \in U = \{u \mid C_u u \le c_u\} \tag{3.41}$$

where X and U are polyhedral and polytopic sets respectively and both contain the origin as an interior point and where $C_x \in \mathbb{R}^{q_x \times n}$ and $c_x \in \mathbb{R}^{q_x}$ and $C_u \in \mathbb{R}^{q_u \times m}$ and $c_u \in \mathbb{R}^{q_u}$. We illustrate that in this case, one can formulate a linear programming problem, similar to the minimization problem $\mathbb{P}_k$, whose feasibility establishes existence of a robustly controlled invariant set $F_k(\mathbf{M}_k)$, i.e. $x \in X$ and $u(x) \in U$ and $Ax + Bu(x) \oplus W \subseteq F_k(\mathbf{M}_k)$ for all $x \in F_k(\mathbf{M}_k)$. Before proceeding recall that, the control law $u(\cdot)$ is defined by (3.10) and (3.12):

$$u(x) = M_{k-1}w_0^0(x) + M_{k-2}w_1^0(x) + \dots + M_1 w_{k-2}^0(x) + M_0 w_{k-1}^0(x),\ \forall x \in F_k(\mathbf{M}_k) \tag{3.42}$$

In order to ensure satisfaction of the state and control constraints (3.41) we impose the following constraints:

$$x \in \alpha X,\ u(x) \in \beta U,\ \forall x \in F_k(\mathbf{M}_k) \tag{3.43}$$

where $(\alpha, \beta) \in (0,1) \times (0,1)$. This constraints can be written in terms of set inclusions as follows:

$$F_k(\mathbf{M}_k) \subseteq \alpha X,\ U(\mathbf{M}_k) \subseteq \beta U,\ U(\mathbf{M}_k) \triangleq M_{k-1}W \oplus \dots \oplus M_1 W \oplus M_0 W \tag{3.44}$$

where $\alpha X = \{x \mid C_x x \le \alpha c_x\}$ and $\beta U = \{u \mid C_u u \le \beta c_u\}$.

By Corollary 1 we have the following:

Corollary 2. *Let $C \in \mathbb{R}^{q \times m}$ and $M \in \mathbb{R}^{m \times n}$ be two matrices and let $W = \{w = Ed + f \mid |d|_\infty \le \eta\}$ and $E \in \mathbb{R}^{n \times t}$ and $f \in \mathbb{R}^n$. Then:*

$$\max_{w \in W} CMw = \eta \text{abs}(CME)\mathbf{1}_t + CMf \tag{3.45}$$

where the maximization is taken row-wise. Moreover, there exists a matrix $T \in \mathbb{R}^{q \times t}$ such that:

$$-T \le CME \le T \tag{3.46}$$

where the inequality is element-wise, and the solution of (3.45) *is:*

$$\max_{w \in W} CMw = \eta T \mathbf{1}_t + CMf \tag{3.47}$$

Let now:

$$\bar{\Omega} \triangleq \{(\mathbf{M}_k, \alpha, \beta, \delta) \mid \mathbf{M}_k \in \mathbb{M}_k,\ F_k(\mathbf{M}_k) \subseteq \alpha X,\ U(\mathbf{M}_k) \subseteq \beta U,$$
$$(\alpha, \beta) \in (0,1) \times (0,1), \alpha + \beta \le \delta\} \tag{3.48}$$

where $F_k(\mathbf{M}_k)$ is given by (3.17) and $U(\mathbf{M}_k)$ by (3.44).

Consider the following minimization problem:

$$\bar{\mathbb{P}}_k : (\mathbf{M}_k^0, \alpha^0, \beta^0, \delta^0) = \arg \min_{\mathbf{M}_k, \alpha, \beta, \delta}\{\delta \mid (\mathbf{M}_k, \alpha, \beta, \delta) \in \bar{\Omega}\} \tag{3.49}$$

Proposition 5. *The minimization problem $\bar{\mathbb{P}}_k$ is a linear programming problem.*

The Proof of this results follows the same arguments emphasized in the proof of Proposition 4 and uses Corollary 2.

Remark 8. It follows from Corollary 2 that in this case a matrix $\Theta_k \triangleq \{T_0, T_1, \dots T_{k-1}\}$ has to be introduced to reflect the inclusion corresponding to hard control constraints $U(\mathbf{M}_k) \subseteq \beta U$ into the set of linear inequalities similarly to (3.37)–(3.38). Since

$$\begin{aligned}
\max_{x \in F_k(\mathbf{M}_k)} C_u u(x) = & \max_{w_0 \in W} C_u[\mathbf{0}\ \mathbf{0}\ \dots\ \mathbf{0}\ I]\mathbf{M}_k w_0 \\
& + \max_{w_1 \in W} C_u[\mathbf{0}\ \mathbf{0}\ \dots\ I\ \mathbf{0}]\mathbf{M}_k w_1 \\
& + \dots \\
& + \max_{w_{k-2} \in W} C_u[\mathbf{0}\ I\ \dots\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k w_{k-2} \\
& + \max_{w_{k-1} \in W} C_u[I\ \mathbf{0}\ \dots\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\mathbf{M}_k w_{k-1}
\end{aligned}$$

It follows by Corollary 2 that there exists $\Theta_k \triangleq \{T_0, T_1, \dots, T_{k-1}\}$ such that

$$\max_{x \in F_k(\mathbf{M}_k)} C_u u(x) = \sum_{i=0}^{k-1} (\eta T_i \mathbf{1}_t + C_u S_i \mathbf{M}_k f) \tag{3.50}$$

where the maximization is taken row-wise and each $T_i \in \mathbb{R}^{q_u \times t}$ satisfies:

$$-T_i \leq C_u S_i \mathbf{M}_k E \leq T_i,\ i = 0, 1, \dots k-1 \tag{3.51}$$

where the inequality is element-wise and S_i is appropriate selection matrix of the form $S_i = [\mathbf{0}\ \mathbf{0}\ \dots\ I\ \dots\ \mathbf{0}\ \mathbf{0}]$.

Remark 9. Let $\gamma \triangleq (\text{vec}(\mathbf{M}_k), \text{vec}(\Lambda_k), \text{vec}(\Theta_k), \alpha, \beta, \delta)$ and let:

$$\begin{aligned}
\bar{\Gamma} \triangleq \{\gamma \mid \mathbf{M}_k \in \mathbb{M}_k,\ & \sum_{i=0}^{k-1} (\eta L_i \mathbf{1}_t + C_x(A^i + D_i \mathbf{M}_k) f) \leq \alpha c_x, \\
& - L_i \leq C_x(A^i + D_i \mathbf{M}_k) E \leq L_i,\ i = 0, 1, \dots k-1, \\
& \sum_{i=0}^{k-1} (\eta T_i \mathbf{1}_t + C_u S_i \mathbf{M}_k f) \leq \beta c_u, \\
& - T_i \leq C_u S_i \mathbf{M}_k E \leq T_i,\ i = 0, 1, \dots k-1, \\
& (\alpha, \beta) \in (0,1) \times (0,1),\ q_a \alpha + q_b \beta \leq \delta\}
\end{aligned} \tag{3.52}$$

where q_a and q_b are weights corresponding to the contraction of the state and control constraints. The minimization problem $\bar{\mathbb{P}}_k$ is then readily transformed into a linear programming problem:

$$\bar{\mathbb{P}}_k : \quad \min_{\gamma} \{\delta \mid \gamma \in \bar{\Gamma}\} \tag{3.53}$$

The following result is a consequence of the discussion above.

Proposition 6. *There exists a robustly controlled invariant set $F_k = F_k(\mathbf{M}_k^0)$ and corresponding control law $u(\cdot)$ defined by* (3.12) *and* (3.10) *with $\mathbf{M}_k = \mathbf{M}_k^0$ satisfying the state and control constraints if and only if the minimization problem $\bar{\mathbb{P}}_k$ defined by* (3.53) *and* (3.52) *is feasible.*

3.3 Relaxing Condition (3.5)

Condition (3.5) can be relaxed as follows. Recall that $W = \{w = Ed + f \mid |d|_\infty \leq \eta\}$ where $E \in \mathbb{R}^{n \times t}$ and $f \in \mathbb{R}^n$ and suppose that the polytopic description of is $W = \{w \mid C_w w \leq c_w\}$ where $C_w \in \mathbb{R}^{q_w \times n}$ and $c_w \in \mathbb{R}^{q_w}$. Furthermore, it is assumed that the origin is an interior point of W. The condition (3.5) can be replaced by the following condition:

$$(A^k + [A^{k-1}B\ A^{k-2}B\ \dots\ AB\ B]\mathbf{M}_k)W \subseteq \varphi W \tag{3.54}$$

where $\varphi \in [0, 1)$, or equivalently in terms of support functions:

$$\max_{w \in W} C_w(A^k + [A^{k-1}B\ A^{k-2}B\ \dots\ AB\ B]\mathbf{M}_k)w \leq \varphi c_w \tag{3.55}$$

where the maximization is taken row-wise. In this case, we consider the set $F_{(\varphi,k)}$ defined by:

$$F_{(\varphi,k)} \triangleq (1-\varphi)^{-1} F_k \tag{3.56}$$

We can establish some properties of the set $F_{(\varphi,k)}$.

Proposition 7. *There exists a control law $u : F_{(\varphi,k)} \mapsto \mathbb{R}^m$ such that $Ax + Bu(x) \oplus W \subseteq F_{(\varphi,k)}$, $\forall x \in F_{(\varphi,k)}$, i.e. the set $F_{(\varphi,k)}$ is robustly controlled invariant for system* (3.1).

Proof. Let x be an arbitrary element of $F_{(\varphi,k)}$. Since $x \in F_{(\varphi,k)}$ it follows by definition of the set $F_{(\varphi,k)}$:

$$\begin{aligned} x =& (1-\varphi)^{-1}(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_0 \\ &+ \ldots + (1-\varphi)^{-1}(A^2 + ABM_0 + BM_1)w_{k-3} + (1-\varphi)^{-1}(A + BM_0)w_{k-2} + (1-\varphi)^{-1}w_{k-1} \end{aligned} \tag{3.57}$$

for some $w_i \in W, i = 0, 1, \ldots k-1$. The last equation can be rewritten in matrix form as:

$$x = D\mathbf{w}, \tag{3.58}$$

for some matrix D easily constructed from (3.57) and the vectorized disturbance sequence $\{(1-\varphi)^{-1}w_0, (1-\varphi)^{-1}w_1, \ldots (1-\varphi)^{-1}w_{k-1}\}$. Let $\mathbf{W}^k \triangleq (1-\varphi)^{-1}W \times (1-\varphi)^{-1}W \times \ldots \times (1-\varphi)^{-1}W$ and for all $x \in F_{(\varphi,k)}$ let $\mathbf{w}^0(x)$ be the unique solution of the following quadratic program:

$$\mathbb{P}_w(x): \quad \mathbf{w}^0(x) = \arg\min_{\mathbf{w}} \{|\mathbf{w}|^2 \mid \mathbf{w} \in \mathbf{W}^k,\ D\mathbf{w} = x\}, \tag{3.59}$$

Hence, $\mathbf{w}^0(x) = \{(1-\varphi)^{-1}w_0^0(x), (1-\varphi)^{-1}w_1^0(x), \ldots (1-\varphi)^{-1}w_{k-1}^0(x)\}$ and since $x \in F_{(\varphi,k)}$ it follows that:

$$\begin{aligned} x =& (1-\varphi)^{-1}(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_0^0(x) \\ &+ \ldots + (1-\varphi)^{-1}(A^2 + ABM_0 + BM_1)w_{k-3}^0(x) + (1-\varphi)^{-1}(A + BM_0)w_{k-2}^0(x) + (1-\varphi)^{-1}w_{k-1}^0(x) \end{aligned} \tag{3.60}$$

Let the control law $u(\cdot)$ be defined by:

$$u(x) \triangleq (1-\varphi)^{-1}M_{k-1}w_0^0(x) + (1-\varphi)^{-1}M_{k-2}w_1^0(x) + \ldots + (1-\varphi)^{-1}M_1 w_{k-2}^0(x) + (1-\varphi)^{-1}M_0 w_{k-1}^0(x) \tag{3.61}$$

where $M_i,\ i = 0, 1, \ldots k-1$ satisfy (3.54). If $x \in F_{(\varphi,k)}$, then:

$$\begin{aligned} x^+ =& Ax + Bu(x) + w \\ =& (1-\varphi)^{-1}(A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \ldots + ABM_{k-2} + BM_{k-1})w_0^0(x) \\ &+ (1-\varphi)^{-1}(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \ldots + ABM_{k-3} + BM_{k-2})w_1^0(x) \\ &+ \ldots + (1-\varphi)^{-1}(A^2 + ABM_0 + BM_1)w_{k-2}^0(x) + (1-\varphi)^{-1}(A + BM_0)w_{k-1}^0(x) + w \end{aligned} \tag{3.62}$$

where each $w_i^0(x) \in W, i = 0, 1, \dots k-1$ by construction and $w \in W$ is arbitrary. It follows that:

$$\begin{aligned} x^+ \in & (1-\varphi)^{-1}(A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \dots + ABM_{k-2} + BM_{k-1})W \\ & \oplus (1-\varphi)^{-1}(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \dots + ABM_{k-3} + BM_{k-2})W \\ & \oplus \dots \oplus (1-\varphi)^{-1}(A^2 + ABM_0 + BM_1)W \oplus (1-\varphi)^{-1}(A + BM_0)W \oplus W \end{aligned} \tag{3.63}$$

but $M_i,\ i = 0, 1, \dots k-1$ satisfy (3.54) so that

$$(A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \dots + ABM_{k-2} + BM_{k-1})W \subseteq \varphi W$$

and consequently

$$(1-\varphi)^{-1}(A^k + A^{k-1}BM_0 + A^{k-2}BM_1 + \dots + ABM_{k-2} + BM_{k-1})W \subseteq \varphi(1-\varphi)^{-1}W$$

Thus,

$$\begin{aligned} x^+ \in & (1-\varphi)^{-1}\varphi W \oplus (1-\varphi)^{-1}(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \dots + ABM_{k-3} + BM_{k-2})W \\ & \oplus \dots \oplus (1-\varphi)^{-1}(A^2 + ABM_0 + BM_1)W \oplus (1-\varphi)^{-1}(A + BM_0)W \oplus W \end{aligned} \tag{3.64}$$

Since $(1-\varphi)^{-1}\varphi W \oplus W = (1-\varphi)^{-1}W$ it follows that:

$$\begin{aligned} x^+ \in & (1-\varphi)^{-1}(A^{k-1} + A^{k-2}BM_0 + A^{k-3}BM_1 + \dots + ABM_{k-3} + BM_{k-2})W \\ & \oplus \dots \oplus (1-\varphi)^{-1}(A^2 + ABM_0 + BM_1)W \oplus (1-\varphi)^{-1}(A + BM_0)W \oplus (1-\varphi)^{-1}W \end{aligned} \tag{3.65}$$

so that $x^+ = Ax + Bu(x) + w \in F_{(\varphi,k)}$ for all $w \in W$. It follows that $Ax + Bu(x) \oplus W \subseteq F_{(\varphi,k)}$ for all $x \in F_{(\varphi,k)}$ with $u(x)$ defined by (3.61) and (3.59). □

It follows from Corollary 1 that condition (3.54) is affine in $\mathbf{M}_k$ because:

$$\begin{aligned} & \max_{w \in W} C_w(A^k + [A^{k-1}B\ A^{k-2}B\ \dots\ AB\ B]\mathbf{M}_k)w = \\ & \max_{w \in W} C_w(A^k + D_k\mathbf{M}_k)w = \eta \text{abs}(C_w(A^k + D_k\mathbf{M}_k)E)\mathbf{1}_t + C_w(A^k + D_k\mathbf{M}_k)f \end{aligned} \tag{3.66}$$

where the maximization is taken row-wise, so that

$$\max_{w \in W} C_w(A^k + [A^{k-1}B\ A^{k-2}B\ \dots\ AB\ B]\mathbf{M}_k)w = \eta Y \mathbf{1}_t + C_w(A^k + D_k\mathbf{M}_k)f \tag{3.67}$$

where $Y \in \mathbb{R}^{q_w \times t}$ and:

$$-Y \leq C_w(A^k + [A^{k-1}B\ A^{k-2}B\ \dots\ AB\ B]\mathbf{M}_k)E \leq Y \tag{3.68}$$

where the inequality is element-wise.

If there are constraints (3.41) imposed on system(3.1) we can formulate the linear programming problem for establishing the existence of the set $F_{(\varphi,k)} = F_{(\varphi,k)}(\mathbf{M}_k)$ that is constraint admissible, as we illustrate next.

We note that

$$U_{(\varphi,k)}(\mathbf{M}_k) = (1-\varphi)^{-1}U(\mathbf{M}_k) \tag{3.69}$$

and

$$F_{(\varphi,k)}(\mathbf{M}_k) = (1-\varphi)^{-1}F_k(\mathbf{M}_k) \tag{3.70}$$

where $F_k(\mathbf{M}_k)$ and $U(\mathbf{M}_k)$ are given by (3.17) and (3.44) respectively. We require the following set inclusions to hold:

$$U_{(\varphi,k)}(\mathbf{M}_k) \subseteq U,\ F_{(\varphi,k)}(\mathbf{M}_k) \subseteq X \tag{3.71}$$

The last equation is equivalent to:

$$(1-\varphi)^{-1}U(\mathbf{M}_k) \subseteq U,\ (1-\varphi)^{-1}F_k(\mathbf{M}_k) \subseteq X \tag{3.72}$$

Since X, U, $F_k(\mathbf{M}_k)$ and $U(\mathbf{M}_k)$ are convex sets, each containing the origin as an interior point, we have that (3.72) is equivalent to:

$$U(\mathbf{M}_k) \subseteq (1-\varphi)U,\ F_k(\mathbf{M}_k) \subseteq (1-\varphi)X \tag{3.73}$$

In order to ensure satisfaction of (3.71) we require:

$$U(\mathbf{M}_k) \subseteq \beta U,\ F_k(\mathbf{M}_k) \subseteq \alpha X,\ \beta \le 1-\varphi,\ \alpha \le 1-\varphi \tag{3.74}$$

and as before $(\alpha, \beta, \varphi) \in (0,1) \times (0,1) \times [0,1)$. Finally, we are ready to formulate a linear programming problem similar to the one in Remark 9.

Remark 10. Let $\gamma \triangleq (\mathrm{vec}(\mathbf{M}_k), \mathrm{vec}(\Lambda_k), \mathrm{vec}(Y), \mathrm{vec}(\Theta_k), \alpha, \beta, \delta, \varphi)$ and let:

$$\begin{aligned}\bar{\Gamma} \triangleq \{\gamma \mid\ & \sum_{i=0}^{k-1}(\eta L_i \mathbf{1}_t + C_x(A^i + D_i\mathbf{M}_k)f) \le \alpha c_x,\\ & -L_i \le C_x(A^i + D_i\mathbf{M}_k)E \le L_i,\ i = 0, 1, \ldots k-1,\\ & \sum_{i=0}^{k-1}(\eta T_i \mathbf{1}_t + C_u S_i \mathbf{M}_k f) \le \beta c_u,\\ & -T_i \le C_u S_i \mathbf{M}_k E \le T_i,\ i = 0, 1, \ldots k-1,\\ & \eta Y \mathbf{1}_t + C_w(A^k + D_k\mathbf{M}_k)f \le \varphi c_w,\\ & -Y \le (C_w(A^k + D_k)\mathbf{M}_k)E \le Y,\\ & (\alpha, \beta, \varphi) \in (0,1) \times (0,1) \times [0,1),\\ & \alpha + \varphi \le 1,\ \beta + \varphi \le 1,\ q_a\alpha + q_b\beta + q_p\varphi \le \delta\}\end{aligned} \tag{3.75}$$

where q_a, q_b and q_p are weights corresponding to the contraction of the state, control constraints and the disturbance polytope. The minimization problem $\bar{\mathbb{P}}_k$ we consider is then readily transformed into a linear programming problem:

$$\bar{\mathbb{P}}_k : \quad \min_{\gamma}\{\delta \mid \gamma \in \bar{\Gamma}\} \tag{3.76}$$

The following result is a consequence of the discussion above.

Proposition 8. *There exists a robustly controlled invariant set $F_{(\varphi,k)} = (1-\varphi)^{-1}F_k(\mathbf{M}_k^0)$ and corresponding control law $u(\cdot)$ defined by* (3.59) *and* (3.60) *with $\mathbf{M}_k = \mathbf{M}_k^0$ satisfying the state and control constraints if and only if the minimization problem $\bar{\mathbb{P}}_k$ defined by* (3.75) *and* (3.76) *is feasible.*

Remark 11 *(Generalization of Optimized Robust Controlled Invariance).* The above results can be extended to the case when the disturbance polytope is an arbitrary polytope $W \triangleq \{w \in \mathbb{R}^n \mid C_w w \le c_w\}$ that contains the origin in its interior. It is also possible to extend the results to the case when the considered norm is $p = 2$. The details will be provided elsewhere.

Remark 12. We note that one can easily modify the cost function that is penalized. For instance, an appropriate choice is positively weighted two norm of the decision variable γ that yields a unique solution, since in this case problem becomes quadratic programming problem of the form

$$\min_{\gamma}\{|\gamma|_Q^2 \mid \gamma \in \bar{\Gamma}\},$$

where Q is positive definite and it represents appropriate weights.

Corollary 3. *Suppose that the minimization problem $\bar{\mathbb{P}}_k$ defined in* (3.75) *and* (3.76) *is feasible for some $k \in \{0, 1, 2, \ldots\}$ and the optimal value of δ_k is δ_k^0 then for all integers $s \ge k$ the minimization problem $\bar{\mathbb{P}}_s$, where k is replaced with s in* (3.75) *and* (3.76)*, is also feasible and the corresponding optimal value of δ_s satisfies $\delta_s^0 \le \delta_k^0$.*

3.4 Comparison with Existing Methods

In order to demonstrate advantages of our method over existing methods briefly reviewed in Section 2, we modify the results of Theorem 1 in [KM03b, KM04] and proceed as follows. Let

$$\mathbb{K} \triangleq \{K \in \mathbb{R}^{m\times n} \mid \quad | \lambda_{\max}(A+BK)| < 1\} \tag{3.77}$$

where $\lambda_{\max}(A)$ denotes the largest eigenvalue of the matrix A. Let $K \in \mathbb{K}$ and let:

$$D_\infty(K) = \lim_{i\to\infty} D_i(K) \tag{3.78}$$

where $D_i(K)$ is defined by (2.2):

$$D_i(K) \triangleq (A+BK)D_{i-1}(K) \oplus W,\ i \geq 1 \text{ and } D_0(K) \triangleq \{0\} \tag{3.79}$$

Since, as is well known that generally $D_\infty(K)$ is impossible to compute , let:

$$D_{(\zeta(K),s(K))}(K) \triangleq (1-\zeta(K))^{-1} D_{s(K)}(K) \tag{3.80}$$

where $D_{s(K)}(K)$ is defined by (3.79) so that:

$$D_{s(K)}(K) = \bigoplus_{i=0}^{s(K)-1} (A+BK)^i W \tag{3.81}$$

So that $D_{(\zeta(K),s(K))}(K)$ is the best possible invariant approximation[1] of the minimal robustly positively invariant set $D_\infty(K)$. We remark that the couple $(\zeta(K), s(K)) \in [0,1) \times \{0,1,2,\ldots\}$ is such that the following set inclusion holds true:

$$(A+BK)^{s(K)} W \subseteq \zeta(K) W \tag{3.82}$$

Let:

$$\mathcal{K} \triangleq \{K \in \mathbb{K} \mid D_{(\zeta(K),s(K))}(K) \subseteq \alpha X,\ KD_{(\zeta(K),s(K))}(K) \subseteq \beta U,\ (\alpha,\beta) \in (0,1)\times(0,1)\} \tag{3.83}$$

where X and U are state and control constraints and where it assumed that $D_{(\zeta(K),s(K))}(K)$ is the best possible approximation of $D_\infty(K)$ in view of discussion above (See (3.80)–(3.82)). Given any $K \in \mathcal{K}$ let

$$\begin{aligned}
\alpha(K) &\triangleq \min_{\alpha\in(0,1)} \{\alpha \mid D_{(\zeta(K),s(K))}(K) \subseteq \alpha X\} \\
\beta(K) &\triangleq \min_{\beta\in(0,1)} \{\beta \mid KD_{(\zeta(K),s(K))}(K) \subseteq \beta U\}
\end{aligned} \tag{3.84}$$

Given any $s \in \{0,1,\ldots\}$ let

$$\mathbf{K}_s = [K' \ K(A+BK)' \ \ldots \ K(A+BK)^{s-2'} \ K(A+BK)^{s-1'}]' \tag{3.85}$$

and note that for any integer $k \leq s$:

$$(A+BK)^k = A^k + [A^{k-1}B \ A^{k-2}B \ \ldots \ AB \ B \ \mathbf{0} \ \ldots \ \mathbf{0}]\mathbf{K}_s \tag{3.86}$$

Let also:

$$\begin{aligned}
\bar{\Omega}_k \triangleq \{(\mathbf{M}_k, \alpha, \beta, \delta, \varphi) \mid \ & F_k(\mathbf{M}_k) \subseteq \alpha X,\ U(\mathbf{M}_k) \subseteq \beta U \\
& (A^k + [A^{k-1}B \ A^{k-2}B \ \ldots \ AB \ B]\mathbf{M}_k)W \subseteq \varphi W, \\
& (\alpha,\beta,\varphi) \in (0,1)\times(0,1)\times[0,1), \\
& \alpha+\varphi \leq 1,\ \beta+\varphi \leq 1,\ q_a\alpha + q_b\beta + q_p\varphi \leq \delta\}
\end{aligned} \tag{3.87}$$

where $F_k(\mathbf{M}_k)$ and $U(\mathbf{M}_k)$ are defined in (3.69) and (3.70).

We can now state the following result that follows easily from the discussion above:

[1]The term *'the best possible invariant approximation'* is used somehow loosely and is related to the finite arithmetic precision involved in the corresponding computations.

Proposition 9. *Let $K \in \mathcal{K}$, where $\mathcal{K}$ is defined in* (3.83)*, and the couple* $(\zeta(K), s(K)) \in [0,1) \times \{0,1,2,\dots\}$ *satisfies* (3.82)*. Then* $(\mathbf{K}_{s(K)}, \alpha(K), \beta(K), q_a\alpha(K)+q_b\beta(K)+q_p\zeta(K), \zeta(K))$ *satisfies* $(\mathbf{K}_{s(K)}, \alpha(K), \beta(K), q_a\alpha(K)+q_b\beta(K)+q_p\zeta(K), \zeta(K)) \in \bar{\Omega}_{s(K)}$ *where* $\bar{\Omega}_k$ *is defined in* (3.87)*.*

The proof of this results follows from straight–forward verification that $(\mathbf{K}_{s(K)}, \alpha(K), \beta(K), q_a\alpha(K)+q_b\beta(K)+q_p\zeta(K), \zeta(K)) \in \bar{\Omega}_{s(K)}$.

Remark 13. Proposition 9 and Corollary 3 imply that for any $K \in \mathcal{K}$ and for all $s \geq s(K)$ the minimization problem $\bar{\mathbb{P}}_s$

$$\bar{\mathbb{P}}_s : (\mathbf{M}_s^0, \alpha^0, \beta^0, \delta^0, \varphi^0) = \arg \min_{(\mathbf{M}_k, \alpha, \beta, \delta, \varphi)} \{\delta \mid (\mathbf{M}_k, \alpha, \beta, \delta, \varphi) \in \bar{\Omega}_s\} \tag{3.88}$$

yields δ^0 that is smaller or equal than the value of $q_a\alpha(K)+q_b\beta(K)+q_p\zeta(K)$.

In view of the previous remark we conclude that our method does at least as well as existing methods. However recalling remark 5, it is easy to conclude that our method improves upon existing methods. Moreover, the crucial advantage of our method lies in the fact that it is easy to incorporate hard state and control constraints into the optimization problem (See (3.75)–(3.76)) and obtain a solution by a single optimization problem in contrast to the existing methods of viability theory[Aub91], where the constraints are not handled in the most efficient way and corresponding set computations are done through recursive set calculations. Moreover there exists no efficient method for computation of a $K \in \mathbb{K}$ such that $K \in \mathcal{K}$, apart from straight forward computations and testing whether $K \in \mathcal{K}$. This advantages will be illustrated by an appropriate numerical example[2].

3.5 Comparison – Illustrative Example

In order to illustrate our results we consider the second order systems:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \tag{3.89}$$

with additive disturbance:

$$W \triangleq \left\{ w \in \mathbb{R}^2 \mid |w|_\infty \leq 1 \right\}. \tag{3.90}$$

The following set of hard state and control constraints is required to be satisfied:

$$X = \{x \mid -3 \leq x^1 \leq 1.85,\ -3 \leq x^2 \leq 3,\ x^1 + x^2 \geq -2.2\},\ U = \{u \mid |u| \leq 2.4\} \tag{3.91}$$

where x^i is the i^{th} coordinate of a vector x.

In the first attempt we obtain the closed loop dynamics by applying three various state feedback control laws to a second order double integrator example (3.89):

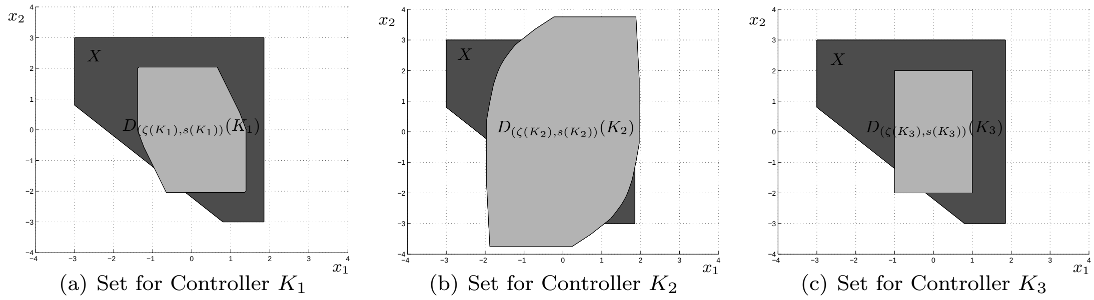
$$\begin{aligned} K_1 &= -[0.72\ 0.98], \\ K_2 &= -[0.96\ 1.24], \\ K_3 &= -[1\ 1] \end{aligned} \tag{3.92}$$

and compute the corresponding sets $D_{(\zeta(K),s(K))}(K)$. The invariant sets $D_{(\zeta(K),s(K))}(K)$ computed by using methods of [Kou02, RKKM03, RKKM04] are shown in Figure 1.

Only the set $D_{(\zeta(K_3),s(K_3))}(K_3)$ is contained in ∞ norm ball $\mathbb{B}_\infty(2)$ but all of the computed sets violate the state constraints as illustrated in Figure 1. We also report that for these state feedback controllers the corresponding control polytopes are:

$$\begin{aligned} U(K_1) &= \{u \mid |u| \leq 2.4680\}, \\ U(K_2) &= \{u \mid |u| \leq 6.4578\}, \\ U(K_3) &= \{u \mid |u| \leq 3\}, \end{aligned} \tag{3.93}$$

[2] In fact, a modification of the optimization problem we consider provides a way for computing $K \in \mathcal{K}$.

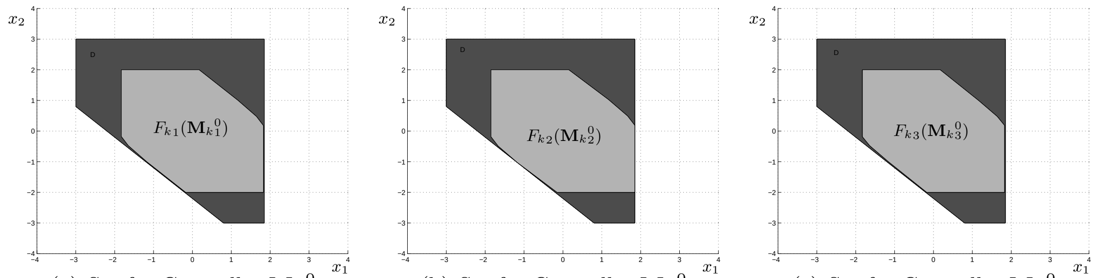


(a) Set for Controller K_1 (b) Set for Controller K_2 (c) Set for Controller K_3

Figure 1: Invariant Approximations of $D_\infty(K_i)$: Sets $D_{(\zeta(K_i),s(K_i))}(K_i),\ i = 1, 2, 3$

where $U(K) \triangleq KD_{(\zeta(K),s(K))}(K)$ so that the control constraints are also violated.

By solving the optimization problem $\bar{\mathbb{P}}_k$ defined in (3.52)–(3.53) we computed the invariant sets $F_{k\,i}(\mathbf{M}_{k\,i}^{\,0}),\ i = 1, 2, 3$ and they are shown in Figure 2.



(a) Set for Controller $\mathbf{M}_{k\,1}^{\,0}$ (b) Set for Controller $\mathbf{M}_{k\,2}^{\,0}$ (c) Set for Controller $\mathbf{M}_{k\,3}^{\,0}$

Figure 2: Invariant Sets $F_{k\,i}(\mathbf{M}_{k\,i}^{\,0}),\ i = 1, 2, 3$

The optimization problem $\bar{\mathbb{P}}_k$ was posed with the following design parameters:

$$\begin{aligned} &k_1 = 5,\ q_\alpha = q_\beta = 1, \\ &k_2 = 5,\ q_\alpha = 0,\ q_\beta = 1, \\ &k_3 = 5,\ q_\alpha = 1,\ q_\beta = 0, \end{aligned} \tag{3.94}$$

The optimization problem $\bar{\mathbb{P}}_k$ yielded the following matrices $\mathbf{M}_{k\,i}^{\,0},\ i = 1, 2, 3$:

$$\begin{aligned} \mathbf{M}_{k\,1}^{\,0} &= \begin{bmatrix} -0.5 & -1 \\ 0.2378 & 0 \\ 0.1139 & 0 \\ 0.0590 & 0 \\ 0.0894 & 0 \end{bmatrix}, \\ \mathbf{M}_{k\,2}^{\,0} &= \begin{bmatrix} -0.4875 & -1 \\ 0.2199 & 0 \\ 0.1154 & 0 \\ 0.0596 & 0 \\ 0.0926 & 0 \end{bmatrix}, \\ \mathbf{M}_{k\,3}^{\,0} &= \begin{bmatrix} -0.5038 & -1 \\ 0.2456 & 0 \\ 0.1132 & 0 \\ 0.0521 & 0 \\ 0.0930 & 0 \end{bmatrix} \end{aligned} \tag{3.95}$$

and the corresponding control polytopes are:

$$\begin{aligned} U(\mathbf{M}_{k1}^{0}) &= \{u \mid |u| \leq 2\}, \\ U(\mathbf{M}_{k2}^{0}) &= \{u \mid |u| \leq 1.9750\}, \\ U(\mathbf{M}_{k3}^{0}) &= \{u \mid |u| \leq 1.9750\}, \end{aligned} \tag{3.96}$$

All the sets constructed from the solution of the optimization problem $\bar{\mathbb{P}}_k$ satisfy state and control constraints as it can be seen from Figure 2 and (3.96). Note that if k was increased there is a possibility that better results would be obtained.

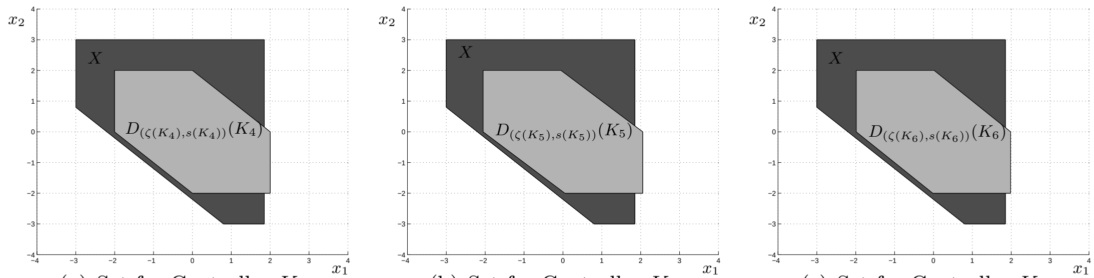
To make comparison in this simple example as fair as possible we consider also the following three state feedback control laws constructed from the first raw of the optimized matrices $\mathbf{M}_k$:

$$\begin{aligned} K_4 &= -[0.5\ 1], \\ K_5 &= -[0.4875\ 1], \\ K_6 &= -[0.5038\ 1] \end{aligned} \tag{3.97}$$

The corresponding sets $D_{(\zeta(K),s(K))}(K)$ are shown in Figure 3. The corresponding control polytopes are:

$$\begin{aligned} U(K_4) &= \{u \mid |u| \leq 2\}, \\ U(K_5) &= \{u \mid |u| \leq 1.975\}, \\ U(K_6) &= \{u \mid |u| \leq 2.076\}, \end{aligned} \tag{3.98}$$

so that the control constraints are satisfied, but unfortunately all of the computed sets violate the state constraints.



(a) Set for Controller K_4 (b) Set for Controller K_5 (c) Set for Controller K_6

Figure 3: Invariant Approximations of $D_\infty(K_i)$: Sets $D_{(\zeta(K_i),s(K_i))}(K_i),\ i = 4, 5, 6$

This simple example and Proposition 9 indicate clear superiority of our method if system is subject to state and control constraints.

4 Application to Robust Optimal Control

Consider the problem of controlling system (3.1) subject to additive disturbances and hard state and control constraints defined in (3.41) over finite horizon of length N and suppose that the additional constraint is:

$$x_N \in X_f \subseteq X \tag{4.1}$$

where X_f is a terminal constraint assumed to be polytopic containing the origin as an interior point and $x_N = \phi(N; x, \pi, \mathbf{w})$, where π is the control policy $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ applied to system (3.1). Solution to this problem requires search over all control policies π belonging to the set $\Pi_N(x)$ defined by:

$$\Pi_N(x) \triangleq \{\pi \mid (\phi(i;x,\pi,\mathbf{w}), \mu_i(\phi(i;x,\pi,\mathbf{w}))) \in X \times U,\ i = 0,1,\dots,N-1, \\ \phi(N;x,\pi,\mathbf{w}) \in X_f,\ \forall \mathbf{w} \in \mathbf{W}\}. \quad (4.2)$$

where $\mathbf{w} = \{w_0, w_1, \dots, w_{N-1}\}$ and $\mathbf{W} = W \times W \times \dots \times W$.

Determination of a control policy $\pi \in \Pi_N(x)$ is usually prohibitively difficult and an appropriate parameterization of the policy π is necessary in order to obtain computationally tractable problem. Recent results in robust optimization [Gus02, BTGGN02] that have been applied to model predictive control in [KM03b, KM04, KA03, KA04, LÖ3b, LÖ3a, vHB03b, vHB03a, vHB02a, vHB02b] showed that an appropriate control policy π can be obtained if feedback laws $\mu_i(\cdot)$ in policy π are parametrized as follows:

$$\mu_0(x) = v_0 \text{ and } \mu_i(x) = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j,\ i \in \{1,2,\dots,N-1\} \quad (4.3)$$

where $\{w_j\},\ j \in \{0,1,2,\dots,N-1\}$ is the actual disturbance realization.

It is shown that with this parametrization one can formulate the convex programming problem, depending on the choice of the cost to be minimized, in which decision variables are the nominal control sequence $\mathbf{v} \triangleq \{v_0, v_1, \dots, v_{N-1}\}$ and the matrix $\mathbf{M}$, specifying the feedback component of the control policy π, defined by:

$$\mathbf{M} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ M_{1,0} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N-2,0} & M_{N-2,1} & \dots & \dots & \mathbf{0} \\ M_{N-1,0} & M_{N-1,1} & \dots & M_{N-1,N-2} & \mathbf{0} \end{bmatrix} \quad (4.4)$$

Properties of this parameterization are studied in more detail in [KM03b, KM04]. We note that the resulting optimization problem is obviously tractable, but demanding since, in the model predictive framework, computation of $\mathbf{v}$ and $\mathbf{M}$ have to be repeated at each sample time. We aim to exploit the results of the previous section to improve computational efficiency and to preserve the desired properties of this parametrization as much as possible.

Our standing assumption is that there exists a matrix $\mathbf{M}_s$ such that the minimization programming problem $\bar{\mathbb{P}}_s$ defined in Remark 10 (or Remark 9) , in which the state constraints are replaced by the terminal set constraints, is feasible for some $s \in \{0,1,2,\dots\}$ yielding the set $F(\varphi, s)$ and the control policy, rendering the set $F(\varphi, s)$ robustly controlled invariant, both defined via matrices $M_i,\ i = 0,1,\dots,s-1$.

Remark 14. We observe that we do not require, in general, that $s = N$ and allow s to be an arbitrary finite integer. Thus s can be also treated as a controller design variable.

Let the set $F(\varphi, s)$ and the matrix $\mathbf{M}_s \triangleq \{M_0, M_1, \dots, M_{s-1}\}$ be constructed from the solution of the minimization programming problem $\bar{\mathbb{P}}_s$ and let, with some abuse of notation:

$$F \triangleq F(\varphi, s) \text{ and } U(F) \triangleq U_{(\varphi,k)}(\mathbf{M}_k) \quad (4.5)$$

where $F(\varphi, s)$ and $U_{(\varphi,k)}(\mathbf{M}_k)$ are defined by (3.69) and (3.70).

Let us define an appropriate reference system that corresponds to the nominal part of the system (3.1) by:

$$z^+ = Az + Bv \quad (4.6)$$

If the initial state is z at time 0 then we denote by $\bar{\phi}(k; z, v)$ the solution to (4.6) at time instant k, given the control sequence $\mathbf{v} \triangleq \{v_0, v_1 \ldots v_{N-1}\}$. We define the set of tighter constraints for system (4.6)

$$Z \triangleq X \ominus F,\ V \triangleq U \ominus U(F) \text{ and } Z_f \triangleq X_f \ominus F \tag{4.7}$$

where $\ominus$ denotes the standard Minkowski (Pontryagin) Set Difference.

Remark 15. The solution of the problem $\bar{\mathbb{P}}_s$ allows for computation of sets Z_f and V by a simple algebraic manipulation and the set Z is easily computed by solving a sequence of linear programming problems. If $X_f = X$ then $Z = Z_f$ and Z and V are obtained by a simple algebraic manipulation.

Let the control action applied to (3.1) be defined by:

$$u = v + t(e) \tag{4.8}$$

where $e \triangleq x - z$ is the error and it satisfies the following difference equation:

$$e^+ = Ae + Bt(e) + w \tag{4.9}$$

Suppose that $e \in F$ and let $t(e)$ be defined by:

$$t(e) \triangleq M_{s-1}w_0^0(e) + M_{s-2}w_1^0(e) + \ldots + M_1 w_{s-2}^0(e) + M_0 w_{s-1}^0(e) \tag{4.10}$$

where, now $\mathbf{w}^0(e) = \{w_0^0(e), w_1^0(e), \ldots, w_{s-1}^0(e)\}$ is the solution of the following quadratic program:

$$\mathbb{P}_w(e): \quad \mathbf{w}^0(e) = \arg\min_{\mathbf{w}}\{|\mathbf{w}|^2 \mid \mathbf{w} \in \mathbf{W}, D\mathbf{w} = e\}, \tag{4.11}$$

where, matrices $M_i,\ i = 0, 1, \ldots, s-1$ and the matrix D and the set $\mathbf{W}$ are constructed as in proof of Proposition 7.

We can now state the following result similar to the one reported in [ML01], but more general:

Proposition 10. *Let $x \in z \oplus F$ and let $u = v + t(e)$, then $x^+ \in z^+ \oplus F$.*

Proof.

$$x^+ = Ax + Bu + w = A(z+e) + B(v + t(e)) + w = Az + Bv + Ae + Bt(e) + w = z^+ + e^+$$

By proposition 7, since $e \in F$, $e^+ \in F$ so that $x^+ \in z^+ \oplus F$. □

Let $\mathcal{V}_N(x)$ be defined as follows:

$$\mathcal{V}_N(x) \triangleq \{(\mathbf{v}, z) \mid (\bar{\phi}(k; z, \mathbf{v}), v_k) \in Z \times V,\ k = 0, 1, \ldots, N-1,\ \bar{\phi}(N; z, \mathbf{v}) \in Z_f,\ x \in z \oplus F\} \tag{4.12}$$

Let the couple $(\mathbf{v}, z)$ be an arbitrary element of $\mathcal{V}_N(x)$ and let the control action applied to system (3.1) at time i be defined by:

$$\mu_i(x_i) = v_i(z) + t(e_i),\ e_i = x_i - z_i \tag{4.13}$$

where $x_i = \phi(i; x, \pi, \mathbf{w})$, $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ and $z_i = \bar{\phi}(i, z, v)$ and $t(\cdot)$ is defined in (4.10) and (4.11).

Proposition 10 allows as to state the following result:

Proposition 11. *Let the couple $(\mathbf{v}, z)$ be an arbitrary element of $\mathcal{V}_N(x)$ and suppose that the control policy $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$, where each $\mu_i(\cdot)$ is defined by (4.13), is applied to actual system (3.1). Then $\phi(i; x, \pi, \mathbf{w}) \in \bar{\phi}(i; z, v) \oplus F$ for all $i = \{0, 1, 2, \ldots, N\}$ and all $\mathbf{w} \in \mathbf{W}$.*

Proof. The proof follows trivially from Proposition 10 by induction. □

Remark 16. Since Z, Z_f and V are defined by (4.7) it follows from Proposition 11 that $\phi(i; x, \pi, \mathbf{w}) \in X$, $\mu_i(\phi(i; x, \pi, \mathbf{w})) \in U$ for all $i = 0, 1, \ldots N-1$ and $\phi(N; x, \pi, \mathbf{w}) \in X_f$.

Remark 17. Computation of the feedback component $t(e)$ can be even further simplified by memorizing the actual disturbances or by solving a corresponding parametric quadratic programming problem in order to obtain explicit solution for the *'optimal'* disturbance sequences used in the feedback component of the control policy π.

The robust optimal control problem we consider is minimization of an appropriate cost function with respect to $(\mathbf{v}, z)$ subject to constraints $(\mathbf{v}, z) \in \mathcal{V}_N(x)$. Appropriate cost function can be defined as follows:

$$V_N(\mathbf{v}, z) \triangleq \sum_{i=0}^{N-1} \ell(z_i, v_i) + V_f(z_N), \tag{4.14}$$

where for all i, $z_i := \bar{\phi}(i; z, \mathbf{v})$ and $\ell(\cdot)$ is the stage cost and $V_f(\cdot)$ is the terminal cost.

The stage cost $\ell(\cdot)$ and the terminal cost $V_f(\cdot)$ can be chosen to be :

$$\ell(x, u) \triangleq \|Qx\|_p + \|Ru\|_p, \; p = 1, 2, \infty \tag{4.15a}$$

$$V_f(x) \triangleq \|Px\|_p, \; p = 1, 2, \infty \tag{4.15b}$$

where P, Q and R are matrices of suitable dimensions. With this choice of the cost function the resulting optimal control problem is standard linear or quadratic programming problem[3]. For instance, if $p = 2$ and $Q > 0$, $P > 0$ and $R > 0$ the resulting optimal control problem is:

$$\mathbb{P}_N(x) : \quad \min_{\mathbf{v}, z} \{V_N(\mathbf{v}, z) \; |(\mathbf{v}, z) \in \mathcal{V}_N(x)\} \tag{4.16}$$

and its unique minimizer is:

$$(\mathbf{v}^0(x), z^0(x)) \triangleq \arg\min_{\mathbf{v}, z} \{V_N(\mathbf{v}, z) \; |(\mathbf{v}, z) \in \mathcal{V}_N(x)\} \tag{4.17}$$

The set of the states such that the optimal control problem $\mathbb{P}_N(x)$ is feasible (the domain of the value function $V_N^0(\cdot)$, the controllability set) is clearly:

$$\mathcal{X}_N \triangleq \text{Proj}_X \mathcal{V}_N(x) \tag{4.18}$$

Remark 18. It is clear that the set sequence $\{\mathcal{X}_i\}$ where each $\mathcal{X}_i \triangleq \text{Proj}_X \mathcal{V}_i(x)$ is a non-decreasing set sequence, i.e. $X_i \subseteq X_{i+1}$ for all i, providing that Z_f is positively invariant for system (4.6).

This is an important observation implying that increase of horizon length enlarges the set of feasible states, since as is well-known the crucial draw-back of some of the proposed *robust* model predictive control schemes in literature is feasibility problems caused by increase of horizon length.

The control action applied to system (3.1) at time $i \in \{0, 1, \ldots, N-1\}$ is then defined by

$$\mu_i(x_i) = v_i^0(x) + t(e_i), \; e_i = x_i - z_i^0(x) \tag{4.19}$$

where $x_i = \phi(i; x, \pi, \mathbf{w})$, and $z_i^0(x) = \bar{\phi}(i, z^0(x), v^0(x))$ and $t(\cdot)$ is defined in (4.10) and (4.11).

One can observe that the robust optimal control problem $\mathbb{P}_N(x)$ is merely a quadratic (or linear, depending on the choice of the cost) programming problem whose dimension exceeds marginally dimension of the corresponding optimal control problem for deterministic case providing clear computational advantage in contrast to the problem in which a matrix $\mathbf{M}$ is also decision variable. Thus computational efficiency is improved drastically. In principle, this advantage is gained by loosing some of the flexibility of the policy defined in (4.3). However, it is quite clear that the robust optimal control problem $\mathbb{P}_N(x)$ is a perfect candidate for efficient robust model predictive control.

[3]Clearly, any other variation of the cost function that yield the standard linear or quadratic programming problem can be also easily implemented within scope of our scheme.

5 Efficient Robust Model Predictive Control

In this section we illustrate how the results of previous sections can be used in order to obtain computationally efficient robust model predictive control of constrained linear discrete time systems subject to additive but bounded disturbances. The scheme we propose is based on implementation of the optimization problem $\mathbb{P}_N(x)$ defined in (4.16) in receding horizon fashion, as is standard in model predictive control. We show that this robust model predictive scheme, if designed appropriately, allows one to establish the robust exponential stability of the set F by exploiting ideas of [MSR], while keeping in mind that Proposition 9 implies that the scheme proposed here has an advantage over the schemes proposed in [MSR, ML01, CRZ01, KRS00, RGK97, KBM96] and is computationally more efficient and simpler for implementation than the schemes proposed in [LCRM04, SM98, KM03b, KM04, KA03, KA04, LÖ3b, LÖ3a, vHB03b, vHB03a, vHB02a, vHB02b].

We will consider the case when the path and the terminal costs are quadratic[4]. Thus Q, R and P in (4.15) are positive definite, and that the terminal cost $V_f(\cdot)$ and the terminal constraint set Z_f satisfy the usual axioms ([MRRS00], page 797), namely:

A1: $Z_f \subset Z$ is positively invariant for $x^+ = A_K x$; $KZ_f \subset V$, where $A_K = A + BK$
A2: $V_f(\cdot)$ is a local control Lyapunov function for the system $x^+ = A_K x$ satisfying $V_f(A_K x) + \ell(x, Kx) \leq V_f(x)$ for all $x \in Z_f$.

where K is and stabilizing state feedback control law, but natural and beneficial choice is of course unconstrained DLQR controller for (A, B, Q, R).

The optimal control problem solved on–line is the problem $\mathbb{P}_N(x)$ defined in (4.16) while the implicit model predictive control law $\kappa_N(\cdot)$, yielded by solution of $\mathbb{P}_N(x)$, is defined by:

$$\kappa_N(x) = v_0^0(x) + t(x - z_0^0(x)) \tag{5.1}$$

where the feedback component $t(\cdot)$ is defined by (4.10)–(4.11).

The main result of this section, can be obtained by a minor modification of Theorem 1 in [MSR]:

Proposition 12. *The set F is robustly exponentially stable for the controlled uncertain system $x^+ = Ax + B\kappa_N(x) + w,\ w \in W$ with region of attraction $\mathcal{X}_N$ defined in* (4.18).

Remark 19. Clearly, this scheme can also be employed to deal with exponentially decaying disturbances as in [MSR].

Further flexibility and computational simplifications can be obtained by considering a set of possible implementations of the robust optimal control problem $\mathbb{P}_N(x)$ defined in (4.16).

Remark 20 *(Flexibility and efficient implementations of robust model predictive control).* The optimization problem $\mathbb{P}_N(x)$ provides an extremely high degree of flexibility that can be appropriately exploited in devising a whole set of variations of the efficient robust model predictive control schemes. It is rather straight forward exercise to devise a set of robust model predictive control schemes , most of which discussed and studied in more detail in [LCRM04], such as: single policy robust model predictive controller, decreasing horizon and variable horizon controller and dual–mode robust model predictive controller. Fuller exposition and rigorous analysis will be provided elsewhere.

6 Numerical Examples

We illustrate the proposed robust model predictive controller by two simple $2-D$ examples.

[4]Bear in mind that similar analysis can be repeated for different (linear, etc.) but appropriate choice of the terminal and path cost.

6.1 Numerical Example 1

The first numerical example is similar to that used in §3.5 except for the fact that the disturbance and constraints are slightly modified. System is double integrator defined in (3.89):

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \tag{6.1}$$

with additive disturbance:

$$W \triangleq \left\{ w \in \mathbb{R}^2 \mid |w|_\infty \leq 0.5 \right\}. \tag{6.2}$$

and the following set of hard state and control constraints is required to be satisfied:

$$X = \{x \in \mathbb{R}^2 \mid x^1 \leq 1.85,\ x^2 \leq 2\},\ U = \{u \mid |u| \leq 2.4\} \tag{6.3}$$

where x^i is the i^{th} coordinate of a vector x. The cost function is defined by (4.14) and (4.15) with $Q = 100I$, $R = 100$; the terminal cost $V_f(x)$ is the value function $(1/2)x'P_f x$ for the optimal unconstrained problem for the nominal system so that

$$P_f = \begin{bmatrix} 194.7123 & 42.2082 \\ 42.2082 & 182.1846 \end{bmatrix} \tag{6.4}$$

The design parameters for the minimization problem $\bar{\mathbb{P}}_s$ (See (3.52)–(3.53)) defining the components of feedback actions of control policy are given by:

$$s = 5,\ q_\alpha = q_\beta = 1, \tag{6.5}$$

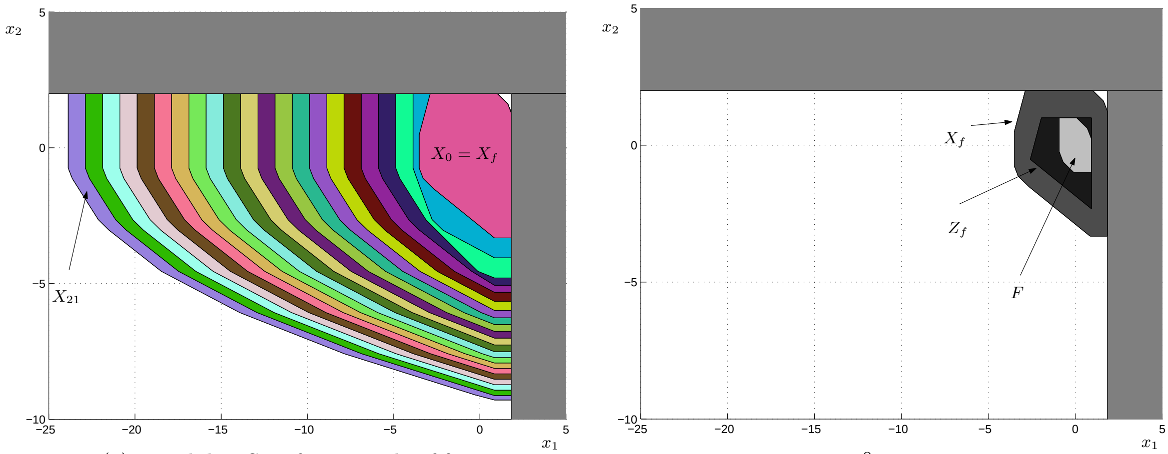
The optimization problem $\bar{\mathbb{P}}_s$ yielded the following matrix $\mathbf{M}_s^0$:

$$\mathbf{M}_s^0 = \begin{bmatrix} -0.3833 & -1 \\ 0 & 0 \\ 0.15 & 0 \\ 0.2333 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.6}$$

and the corresponding invariant set $F = F(\mathbf{M}_s^0)$ shown in Figure 4 together with the terminal set $X_f = Z_f \oplus F$ where Z_f satisfies **A1** and is the maximal positively invariant set for system $z^+ = (A + BK)z$ under the tighter constraints $Z = X \ominus F$ and $V = U \ominus U(F)$ where K is unconstrained DLQR controller for (A, B, Q, R). The sequence of the sets $X_i,\ i = 0, 1, \ldots, 21$, where X_i is the domain of $V_i^0(\cdot)$, is also shown in Figure 4.

The horizon length $N = 21$ is chosen to illustrate clear computational advantage of our approach in contrast to the optimization problems considered in [KM04, KA04, LÖ3b, LÖ3a, vHB03b, vHB03a, vHB02a, vHB02b], that would suffer from dimensionality problems of decision variable since a matrix $\mathbf{M}$ defined in (4.4) is also a decision variable and it increases significantly dimension of decision variable. To be more precise the dimension of decision variable in the optimal control problem $\mathbb{P}_N(x)$ is $21 + 2 = 23$ while the decision variable (that is, in fact, a couple $(\mathbf{v}, \mathbf{M})$) in the above listed methods would have dimension $21 + 2 \cdot 1 \cdot 21 = 63$, and its dimension would significantly increase with the dimension of the state space. Also computation of feedback component is done only once off-line and dimension in this simple example is reasonable lower, in fact $s < N/4$ rendering optimization problem easier to solve.

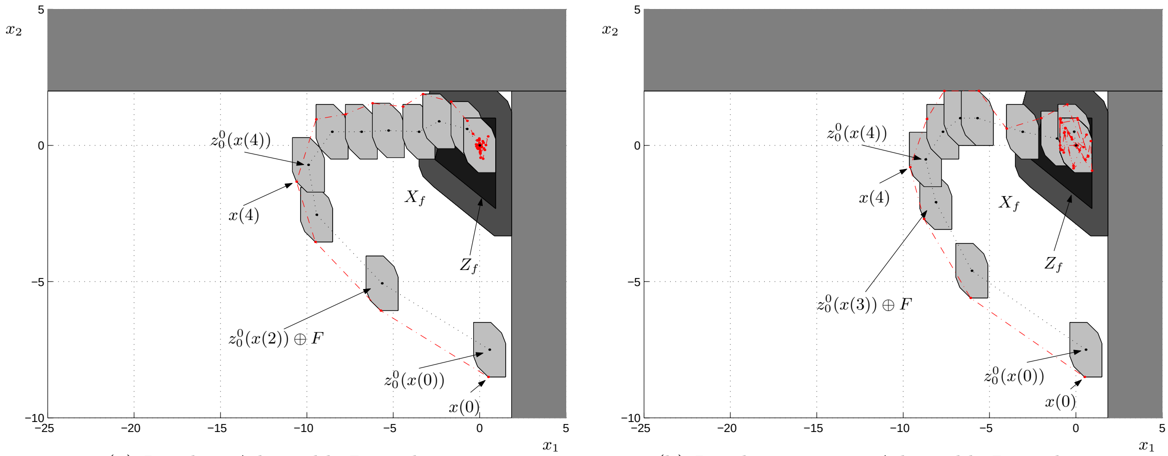
A state trajectory for initial state $x_0 = (0.5, -8.5)'$ is shown in Figure 5 for two cases. The first case illustrates a state trajectory corresponding to a sequence of random, but extreme, disturbances and the second case gives a state trajectory corresponding to a sequence of random admissible disturbances. In both illustrations the dash-dot line is the actual trajectory $\{x(i)\}$ disturbances while the dotted line is the sequence $\{z_0^0(x(i))\}$ of optimal initial states for corresponding reference system.



(a) Feasibility Sets for example of §6.1

(b) Sets $F = F_5(\mathbf{M}_5^0), Z_f$ and $X_f = Z_f \oplus F$

Figure 4: Sets $X_i,\ i = 0, 1, \ldots, 21,\ F = F_5(\mathbf{M}_5^0),\ Z_f$ and $X_f = Z_f \oplus F$



(a) Random Admissible Disturbances

(b) Random Extreme Admissible Disturbances

Figure 5: Trajectories for example of §6.1

6.2 Numerical Example 2

Our second numerical example is a linearized model of a flight vehicle sampled every 0.2 s:

$$x^+ = \begin{bmatrix} 0.9814 & -0.1944 \\ 0.1838 & 0.9386 \end{bmatrix} x + \begin{bmatrix} 0.0272 \\ -0.2694 \end{bmatrix} u + w \tag{6.7}$$

with additive disturbance:

$$W \triangleq \left\{ w \in \mathbb{R}^2 \mid |w|_\infty \leq 0.1 \right\}. \tag{6.8}$$

and the following set of hard state and control constraints is required to be satisfied:

$$X = \{x \in \mathbb{R}^2 \mid x^2 \geq 0.7\},\ U = \{u \mid |u| \leq 2\} \tag{6.9}$$

where x^i is the i^{th} coordinate of a vector x. The cost function is defined by (4.14) and (4.15) with $Q = I$, $R = 1$; the terminal cost $V_f(x)$ is the value function $(1/2)x'P_f x$ for the optimal unconstrained problem for the nominal system so that

$$P_f = \begin{bmatrix} 8.0805 & -1.9366 \\ -1.9366 & 4.8150 \end{bmatrix} \tag{6.10}$$

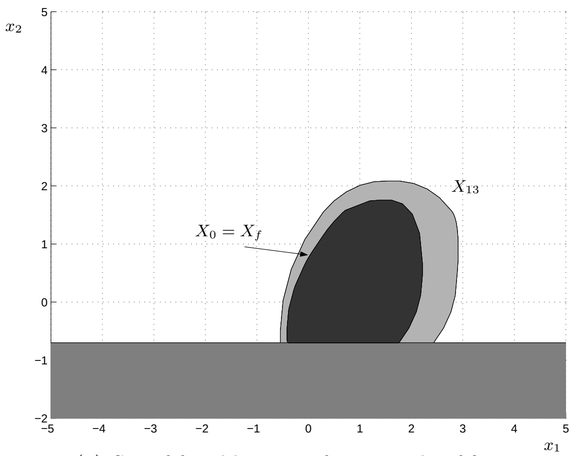
The design parameters for the minimization problem $\bar{\mathbb{P}}_s$ (See (3.52)–(3.53)) defining the components of feedback actions of control policy are given by:

$$s = 5,\ q_\alpha = q_\beta = 1, \tag{6.11}$$

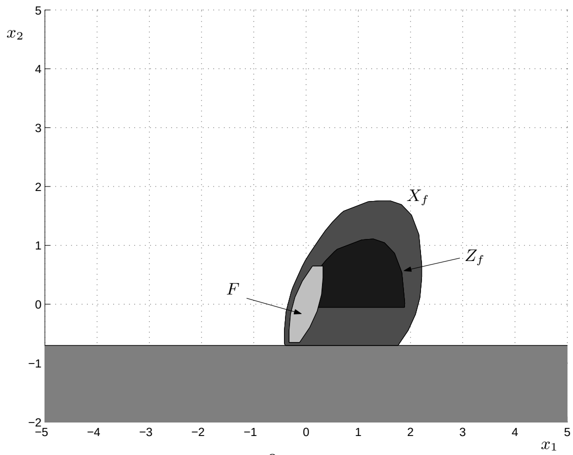
The optimization problem $\bar{\mathbb{P}}_s$ yielded the following matrix $\mathbf{M}_s^0$:

$$\mathbf{M}_s^0 = \begin{bmatrix} -3.6211 & 3.9128 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 4.5578 & -0.4541 \end{bmatrix} \tag{6.12}$$

and the corresponding invariant set $F = F(\mathbf{M}_s^0)$ shown in Figure 6 together with the terminal set $X_f = Z_f \oplus F$ where Z_f satisfies **A1** and is the maximal positively invariant set for system $z^+ = (A + BK)z$ under the tighter constraints $Z = X \ominus F$ and $V = U \ominus U(F)$ where K is unconstrained DLQR controller for (A, B, Q, R). The horizon length is $N = 13$. The set of feasible states X_{13} (the domain of $V_N^0(\cdot)$) is also shown in Figure 6.



(a) Set of feasible states for example of §6.2



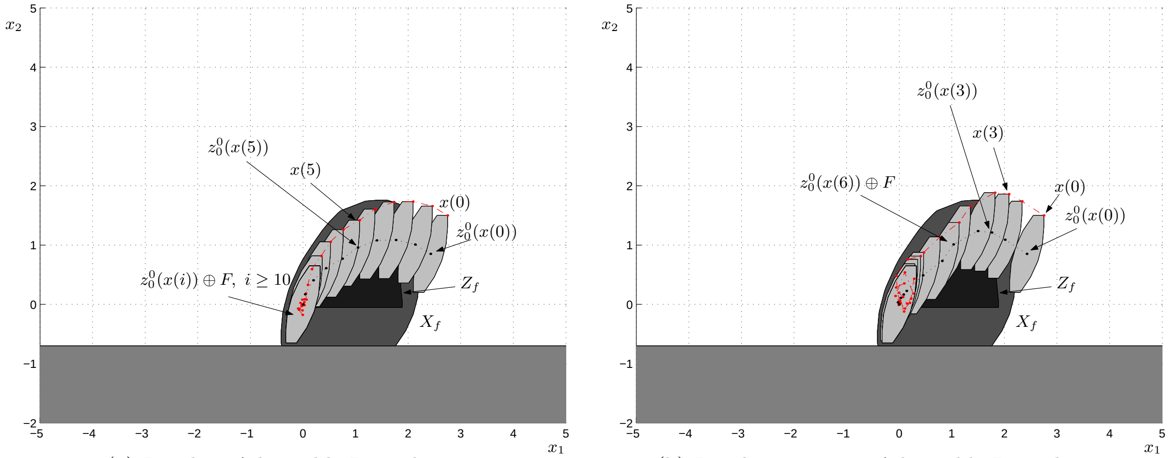
(b) Sets $F = F_5(\mathbf{M}_5^0), Z_f$ and $X_f = Z_f \oplus F$

Figure 6: Sets $X_{13},\ F = F_5(\mathbf{M}_5^0),\ Z_f$ and $X_f = Z_f \oplus F$

A state trajectory for initial state $x_0 = (2.75, 1.5)'$ is shown in Figure 7 for two cases. The first case illustrates a state trajectory corresponding to a sequence of random, but extreme, disturbances and the second case gives a state trajectory corresponding to a sequence of random admissible disturbances. In both illustrations the dash-dot line is the actual trajectory $\{x(i)\}$ disturbances while the dotted line is the sequence $\{z_0^0(x(i))\}$ of optimal initial states for corresponding reference system.

7 Conclusions and Future Research

In this note we introduced the concept of optimized robust controlled invariance. It was shown that an appropriate optimization problem can be posed and that its solution allows for construction of a robustly controlled invariant set contained in the minimal norm ball. The method has an important feature that is handling hard state and control constraints more efficiently than standard recursive set computations of viability theory. These results improve upon existing results for computation of the invariant approximations of minimal robustly positively invariant set for linear discrete time



(a) Random Admissible Disturbances

(b) Random Extreme Admissible Disturbances

Figure 7: Trajectories for example of §6.2

systems. The proposed *optimized robust controlled invariance* algorithms were illustrated by an appropriate example, in which we shown their clear superiority to existing methods.

It is further demonstrated how these results can be used in robust optimal control problem. An interesting feature of the proposed robust optimal control problem is its computational simplicity, in fact the decision variable of the considered robust optimal control problem is of approximately same dimension as the one in corresponding optimal control problem for deterministic case. It was also illustrated how these robust optimal control problem can be used in robust model predictive control of constrained linear discrete time systems subject to bounded but additive disturbances. The proposed scheme is computationally very simple and efficient and allows for a very strong result of robust exponential stability of an *optimized* robustly controlled invariant set to be established. A set of examples is provided to illustrate this efficient robust model predictive control algorithm.

Future research can extend presented results to the case when disturbance belongs to an arbitrary polytope. It is also clear that our results can address some of the issues in the polytopic game problem. Our results would complement result reported in [CD00, Car00] in a sense that appropriate target or *home* sets can be computed by our procedure.

Moreover, it is also possible to extend the results to the case when consider system is parametrically uncertain. Finally, more detailed analysis of the presented robust model predictive scheme will also be presented.

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