# An Hamilton-Jacobi setup for the static output feedback stabilization of nonlinear systems

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February 20, 2002

**Abstract**

The (local) static output feedback stabilization problem for a class of nonlinear (affine) systems is discussed. A sufficient condition is established and a (partial) converse is worked out. This condition provides a counterpart to a well known result of linear systems theory. The effectiveness of the developed theory is illustrated with a simple example.

## 1 Introduction

The static output feedback stabilization (SOF) problem consists in finding, for a given input-output control system, a static output feedback control law such that the resulting closed-loop system is asymptotically stable. Despite the simplicity of this statement and the great amount of literature, the SOF problem is still one of the most celebrated open issues in systems and control. This problem is important in its own right, since static output feedback controllers are less expensive to be implemented and (possibly) more reliable in practice. In particular, static output feedback controllers can be employed as *back-up* controllers, *i.e.* very simple controllers which are not active during the regular operation of the plant, but which are used in case of faults.

Finally, it is easy to show that the design of dynamic output feedback controllers (or of asymptotic state observers) boils down to the solution of a static output feedback control problem for an augmented system.
The SOF stabilization problem for linear systems has been widely studied, see [12] and references therein and the recent contributions [7, 6, 8, 10, 13, 2]. Note however that, even in the case of linear systems, several problems are still open.
The SOF stabilization problem for nonlinear systems has not been studied with similar interest and, to the best of the authors's knowledge, no general result is available. However, several results have been developed for nonlinear passive systems, see *e.g.* [4, 5].
The present paper tries to produce a step forward in the understanding of the local SOF stabilization problem for nonlinear systems, providing a sufficient condition, together with a partial converse, which constitute the counterpart, *mutatis mutandis*, of well-known conditions for linear systems.
The paper is organized as follows. In Section 2 the problem is formally stated, whereas in Section 3 the sufficient and the necessary conditions are provided in terms of the solution of a constrained Hamilton-Jacobi equation along with a rank condition. Finally, Section 4 and Section 5 contain a simple illustrative example, some comments and hints for future research.

## 2 Preliminaries

Consider the nonlinear (affine) system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

where $x \in I\!\!R^n$ denotes the state of the system, $u \in I\!\!R^m$ the control input, $y \in I\!\!R^p$ the measured output, and the mappings $f(x)$, $g(x)$ and $h(x)$ are smooth mappings defined in a neighborhood of the origin of $I\!\!R^n$. Moreover, without loss of generality, we also assume that $x = 0$ is an equilibrium point, *i.e.* $f(0) = 0$ and $h(0) = 0$.
The local *static output feedback stabilization* problem for system (1)-(2) consists in finding, if possible, a static control law described by

$$u = F(y) \tag{3}$$

such that the origin is a locally asymptotically stable equilibrium of the closed-loop system. If such an output feedback does exist, we say that the

system (1)-(2) is SOF *stabilizable* and that $F(\cdot)$ is a solution of the SOF stabilization problem.
Finally, to describe the main results of the paper we introduce the following definitions.

**Definition 1** *The pair $\{f, h\}$ is said to be locally detectable (observable) if there exists a neighborhood $U$ of the point $x = 0$ such that, if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \geq 0$ and $h(x(t)) = 0$ for all $t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$ ($x(t) = 0$ for all $t \geq 0$).*

**Definition 2** *Given a smooth mapping $y = h(x)$, we denote with*[1] $\ker(h)$ *the set of all $x$ such that $y = 0$.*

## 3 Static output feedback stabilization for nonlinear systems

This section contains the main results of the paper, namely two sufficient conditions for the solution of the local SOF stabilization problem for nonlinear systems and a partial converse. It is also shown that the sufficient conditions, once specialized to the case of linear systems, coincide with well-known sufficient (and necessary) conditions proposed in the literature, see [10, 2].

**Theorem 1** *Consider the system (1)-(2) and assume that the pair $\{f, h\}$ is locally detectable and that* $\operatorname{rank}\left(\left.\frac{\partial h}{\partial x}\right|_{x=0}\right) = p$. *Suppose moreover that there exist a scalar function $V(x) \in C^1$, positive definite in a neighborhood of the origin, and a $m \times 1$ matrix function $G(x) \in C^1$ such that*

$$\begin{aligned} 0 &= V_x(x)f(x) - \frac{1}{4}V_x(x)g(x)g'(x)V_x'(x) + h'(x)h(x) + G'(x)G(x) && (4)\\ 0 &= V_x f(x), \qquad \forall x \in \ker(h). && (5) \end{aligned}$$

*Then there exists a orthogonal matrix $T(x) \in C^1$ such that the function*

$$\eta(x) = T(x)G(x) - \frac{1}{2}g'(x)V_x'(x) \tag{6}$$

*is such that*

[1] This is also denoted with $h^{-1}(0)$, see *e.g.* [9].

*(i) for all $x \in \ker(h)$*

$$\eta(x) = 0; \tag{7}$$

*(ii) the origin is a locally asymptotically stable equilibrium of the system $\dot{x} = f(x) + g(x)\eta(x)$;*

*(iii) the trajectories $x(t)$ of the system $\dot{x} = f(x) + g(x)\eta(x)$ starting close to the origin are such that the output $y(t) = h(x(t))$ and the control $u(t) = \eta(x(t))$ are square integrable signals.*

*Moreover, if*

$$p \geq k = \mathrm{rank}\left(\frac{\partial \eta(x)}{\partial x}\right) \qquad \forall x \in \Omega \tag{8}$$

*for some constant $k$ and some neighborhood $\Omega$ of $x = 0$, then*

*(iv) in a neighborhood of the origin, $\eta(x)$ is a function of $y$, i.e. $\eta(x) = \phi(y)$ for some smooth function $\phi(\cdot)$.*

**Proof.** *Point (i).* By equations (4) and (5) it follows

$$\frac{1}{4} V_x(x) g(x) g'(x) V_x'(x) = G'(x) G(x), \quad \forall x \in \ker(h).$$

This means that there exists a smooth orthogonal matrix $T(x)$ such that

$$T(x) G(x) = \frac{1}{2} g'(x) V_x'(x), \quad \forall x \in \ker(h).$$

*Point (ii).* By equation (6) and the orthogonality of the matrix $T(x)$, the Hamilton-Jacobi equation (4) can be rewritten as

$$0 = V_x(x)(f(x) + g(x)\eta(x)) + h'(x)h(x) + \eta'(x)\eta(x). \tag{9}$$

Note now that detectability of the pair $\{f, h\}$ implies the detectability of the pair $\{f + g\eta, \mathrm{col}(h, \eta)\}$. Hence by a classical Lyapunov argument claim (ii) holds.

*Point (iii).* Equation (9) can be rewritten as

$$\dot{V} = -h'(x(t))h(x(t)) - \eta'(x(t))\eta(x(t)).$$

Hence, by local asymptotic stability of the closed-loop system, one has

$$V(x(0)) = \int_0^\infty (\|h(x(t))\|^2 + \|\eta(x(t))\|^2) dt,$$

which establishes the claim.

*Point (iv).* This claim follows from the assumptions, equation (8) and the rank theorem, see [9, Appendix A]. ■

**Remark 1** The existence of a positive definite function $V(x)$ satisfying equation (4), together with the detectability of the pair $\{f, h\}$, implies the existence of state feedback control law locally asymptotically stabilizing system (1). However, unlike the case of linear systems, the existence of a locally stabilizing feedback for system (1), together with the detectability of the pair $\{f, h\}$, does not imply the existence of a (locally) positive definite function $V(x)$ satisfying equation (4).

**Remark 2** Condition (5) is trivially satisfied if $h(x) = x$, *i.e.* if the whole state is available for feedback.

The conditions in Theorem 1 involve two unknown, the function $V(x)$ and the mapping $G(x)$. Conditions involving only one unknown, namely only $V(x)$, can be easily derived as expressed in the following statement.

**Corollary 1** *Assume that there exists a scalar function $W(x) \in C^1$, positive definite in a neighborhood of the origin, such that*

$$
\begin{aligned}
0 &\geq W_x(x)f(x) - \frac{1}{4}W_x(x)g(x)g'(x)W_x'(x) + h'(x)h(x) \qquad (10)\\
0 &= W_x f(x), \quad \forall x \in \ker(h). \qquad (11)
\end{aligned}
$$

*Then there exist a (non-unique) $m \times 1$ matrix function $G(x) \in C^1$ and a scalar function $V(x) \in C^1$, positive definite in a neighborhood of the origin, such that equations (4) and (5) are satisfied.*

**Proof.** The claim is trivially obtained setting $V(x) = W(x)$ and $G(x)$ such that

$$
G'(x)G(x) = -V_x(x)f(x) + \frac{1}{4}V_x(x)g(x)g'(x)V_x'(x) - h'(x)h(x) \geq 0.
$$

■

Unlike linear systems, for which necessary and sufficient conditions are available, for general nonlinear systems only partial converse results can be derived, as expressed in the following statement, which is a partial converse of Theorem 1. Note that a partial converse of Corollary 1 can be obtained with similar arguments.

**Theorem 2** *Consider the system (1)-(2) and assume that the pair* $\{f, h\}$ *is locally observable. Assume moreover that there exists a continuous function* $\phi(y)$*, with* $\phi(0) = 0$*, such that*

*(a) the origin is a locally asymptotically stable equilibrium of the system* $\dot{x} = f(x) + g(x)\phi(y)$*;*

*(b) the trajectories* $x(t)$ *of the system* $\dot{x} = f(x) + g(x)\phi(y)$ *starting close to the origin are such that the output* $y(t) = h(x(t))$ *and the control* $u(t) = \phi(y(t))$ *are square integrable signals.*

*Then there exist a scalar function* $V(x) \in C^1$*, positive definite in a neighborhood* $\Omega$ *of the origin, a* $m \times 1$ *continuous function* $G(x)$*, and an orthogonal matrix* $T(x) \in C^1$ *such that*

$$
\begin{array}{llcl}
(i) & 0 & = & V_x(x)f(x) - \frac{1}{4}V_x(x)g(x)g(x)'V_x(x) + h'(x)h(x) + G'(x)G(x) \\
(ii) & 0 & = & V_x(x)f(x), \quad \forall x \in \ker(h) \\
(iii) & 0 & = & T(x)G(x) - \frac{1}{2}g'(x)V_x(x), \quad \forall x \in \ker(h) \\
(iv) & p & \geq & \mathrm{rank}(\frac{\partial \phi}{\partial x}), \qquad \forall x \in \Omega.
\end{array}
$$

**Proof.** Given an initial condition $\bar{x} \in \Omega$, let $x(t)$ be the corresponding state trajectory of the system $\dot{x} = f(x) + g(x)\phi(y)$ and define

$$V(\bar{x}) = \int_0^\infty \|h(x(t))\|^2 + \|\phi(y(t))\|^2 dt.$$

By assumption (b) and the observability of the pair $\{f, h\}$, we conclude that $V(\bar{x})$ is positive definite and differentiable in $\Omega$. Moreover, it satisfies equation (9) with $\eta(x)$ replaced by $\phi(h(x))$. Set now $G(x) = \phi(h(x)) + \frac{1}{2}g'(x)V_x'$ and $T(x) = I$ and observe that points (i)-(iii) are trivially satisfied. Finally, claim (iv) follows directly from $\phi(y)$ being a function of $y$ only. ■

To conclude the section we show that, for linear systems, the conditions expressed in Theorem 1 are the nonlinear counterpart of well-known sufficient (and necessary) conditions, as given *e.g.* in [10].

**Corollary 2** *Consider the system (1)-(2). Suppose* $f(x) = Ax$*,* $g(x) = B$*,* $h(x) = Cx$*, the pair* $\{A, B\}$ *is controllable and the pair* $\{A, C\}$ *is observable.*

*The system is SOF stabilizable, by means of a linear feedback, if (and only if) there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $G \in \mathbb{R}^{m\times n}$ such that*

$$A'P + PA - PBB'P + C'C + G'G = 0$$

$$V(A'P + PA)V = 0$$

*with* $V = I - C'(CC')^{-1}C$.

**Proof.** The proof of the sufficiency is a trivial consequence of Theorem 1. For the proof of the necessity see [10, 2]. ■

# 4 A simple illustrative example

In this section we present a simple example to illustrate Theorem 1. Consider a rigid body in an inertial reference frame and let $x_1$, $x_2$ and $x_3$ denote the angular momentum components along a body fixed reference frame having the origin at the center of gravity and consisting of the three principal axes. The Euler's equations for the rigid body with two independent controls are

$$\begin{aligned} \dot{x}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) x_2 x_3 + u_1 \\ \dot{x}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) x_3 x_1 + u_2 \\ \dot{x}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) x_1 x_2 + \alpha u_1 + \beta u_2 \end{aligned} \tag{12}$$

where $I_1 > 0$, $I_2 > 0$ and $I_3 > 0$ denote the principal moments of inertia, $u = \mathrm{col}(u_1, u_2)$ are the control torques, and $\alpha$ and $\beta$ are two constant numbers depending on the location of the actuators. The only variable available for feedback is $y = Cx$ where $C$ is a constant matrix which describes the position and the number of the sensing devices.
The problem of stabilization using dynamic feedback (either observer based or not) has been addressed in [1, 11]. We now focus on the static output feedback stabilization problem and we show how it is possible to make use of the result in Theorem 1. The point of departure is the consideration that for system (12) there exist two positive definite functions which are natural candidate to test the conditions in Theorem 1, namely the kinetic energy, *i.e.*

$$K(x) = \frac{1}{2}\left(\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3}\right)$$

and the modulo of the angular momentum vector, *i.e.*

$$P(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right).$$

In fact, simple calculations show that along the trajectories of the system (12) with $u_1 = u_2 = 0$ one has

$$\dot{K} = \dot{P} = 0.$$

As a result, the following facts can be established.

**Proposition 1** *Consider the system (12) with output $y = Cx$. Suppose that*

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix},$$

*that* $\mathrm{rank}(C) = 2$ *and that the system is detectable.*
*The origin can be made a globally asymptotically stable equilibrium of the system by means of static output feedback if there exists a pair $(\lambda, \mu)$ such that*

$$\{(\lambda,\mu) \mid [\lambda,\mu]' \in \mathrm{ker}J\} \cap_{i=1,2,3} \{(\lambda,\mu) \mid \lambda + \frac{\mu}{I_i} > 0\} \neq \emptyset, \tag{13}$$

*where*

$$J = \begin{bmatrix} (\beta C_{12} - C_{13})I_2 I_3 & \beta C_{12} I_2 - C_{13} I_3 \\ (\alpha C_{12} + C_{23})I_1 I_3 & \alpha C_{12} I_1 + C_{23} I_3 \end{bmatrix}$$

*and*

$$C_{ij} = \det \begin{bmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{bmatrix}.$$

**Proof.** If condition (13) holds there exist $\lambda^\star$ and $\mu^\star$ such that the function

$$V(x) = \lambda^\star P(x) + \mu^\star K(x)$$

is positive definite and satisfies condition (5). To conclude the proof it suffices to verify that the function $V(x)$ satisfies condition (10) as well. For, note that with this selection of $V(x)$, condition (10) reduces to

$$C'C \leq XBB'X \tag{14}$$

where $X$ is the (constant) Hessian matrix of the function $V(x)$ and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{bmatrix}.$$

Note now that condition (14) implies

$$B^{\perp}X^{-1}C'CX^{-1}(B^{\perp})' = 0$$

and this holds if $\lambda^{\star}$ and $\mu^{\star}$ are in the kernel of $J$. ∎

**Remark 3** The condition expressed in Proposition 1 can be given a simple geometrical interpretation, as shown in Figure 1. In the space $(\lambda, \mu)$ the set

$$E = \cap_{i=1,2,3}\{(\lambda, \mu) \mid \lambda + \frac{\mu}{I_i} > 0\}$$

is a convex set, *i.e.* the shaded area in Figure 1, whereas the set

$$\mathcal{J} = \{(\lambda, \mu) \in \ker J\}$$

is either a straight line through the origin, if $\det J = 0$, or the singleton $(\lambda, \mu) = (0, 0)$, if $\det J \neq 0$. As a result, the sufficient condition in Proposition 1 could be easily tested. For instance, see [3], typical values for the principal moments of inertia of a small satellite are $I_1 = 27$ kg m$^2$, $I_2 = 17$ kg m$^2$, and $I_3 = 25$ kg m$^2$. Suppose now that $\alpha = 0$ and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \end{bmatrix}.$$

Then, a simple calculation, show that the conditions in Proposition 1 hold if $\beta$ and $\gamma$ are such that

$$\gamma > \frac{17}{108}\beta > 0.$$

**Proposition 2** *Consider the system (12) with output $y = Cx$. Suppose that*

$$C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix},$$

*that* $\mathrm{rank}(C) = 1$ *and that the system is detectable. The origin can be made a globally asymptotically stable equilibrium by means of static output feedback if*

$$\{(\lambda, \mu) \in \Omega\} \cap_{i=1,2,3} \{(\lambda, \mu) \mid \lambda + \frac{\mu}{I_i} > 0\} \neq \emptyset, \tag{15}$$

*where*

$$\Omega = \{(\lambda, \mu) \mid \det\mathcal{J} = 0\}$$

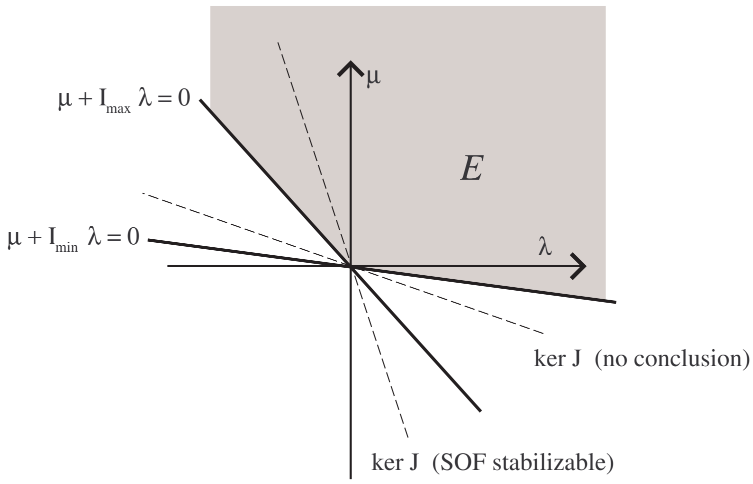


Figure 1: Graphical representation of the sets $E$ and $\mathcal{J}$. If $\mathcal{J} = \ker J$ intersects the shaded region then the stabilization problem is solvable, otherwise the sufficient condition does not yield any conclusion.

*and*

$$\mathcal{J} = \begin{bmatrix} \lambda + \dfrac{\mu}{I_1} & 0 & c_1 \\ 0 & \lambda + \dfrac{\mu}{I_2} & c_2 \\ \alpha(\lambda + \dfrac{\mu}{I_3}) & \beta(\lambda + \dfrac{\mu}{I_3}) & c_3 \end{bmatrix}.$$

**Remark 4** As Proposition 1, also Proposition 2 admits a simple geometrical interpretation. For, note that the set $\Omega$ is either the singleton $(\lambda, \mu) = (0, 0)$, or the line $\lambda = 0$ if $c_2\beta I_2 - c_3 I_3 + \alpha c_1 I_1 = 0$, or the line $\mu = 0$ if $-c_3 + \beta c_2 + \alpha c_1 = 0$, or two (possibly coincident) straight lines through the origin.

# 5 Concluding remarks

The problem of local static output feedback stabilization for nonlinear systems has been studied. A sufficient condition has been developed, together with a partial converse. These are a nonlinear enhancement of existing conditions for linear systems.
These results have been used to derive sufficient conditions for static output

feedback stabilization of the Euler's equations with two controls and with one or two measured outputs. It is an open problem to verify if the conditions expressed in Propositions 1 and 2 are also necessary for static output feedback stabilization.
Note that the results in Propositions 1 and 2 rely on the existence of two conserved quantities, which are natural candidate to test the conditions in Theorems 1 or 2. It is therefore expectable that the proposed theory might be used in the design of static output feedback controllers for general Hamiltonian systems.

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