Practical L_2 disturbance attenuation for nonlinear systems ⋆

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Abstract

In this work, a notion of generalized L_2-gain for nonlinear systems, where the gain is considered as a function of the state instead of a (global) constant, is presented. This new notion seems to be adequate to characterize the gain properties of several nonlinear systems which do not possess a uniform L_2-gain property (i.e. the L_2-gain depends on the operating point). Moreover, a notion of practical L_2-gain attenuation, which extends the standard definition and parallels (mutatae mutandis) the concepts of practical stability, is also proposed.

Key words: Nonlinear systems, Disturbance rejection, H-infinity control, Robust stability.

1 Introduction

The L_2-gain analysis of affine nonlinear systems and the L_2 disturbance attenuation problem have been widely studied in the literature. For a state of the art, see *e.g.*[5,13] for the smooth case, and [7] for the non-differentiable one. All these studies deal with the existence of a solution to a partial differential inequality, which reflects the *dissipative* behavior of the system for a certain supply rate. Although the L_2 paradigm is extremely powerful, for certain systems, such as the system describing the evolution of the angular velocity of a rigid body with two actuators and subject to external disturbances, the L_2 disturbance attenuation problem is not (generically) solvable in the usual sense, see [1]. Therefore, one may wonder how "far" such systems are from obtaining the L_2 disturbance attenuation property.

This paper proposes a definition of *generalized* L_2-gain property, where the L_2-gain constant γ is replaced by $\mu(\cdot)$, a positive function of the state. This new notion of gain stems from the observation that nonlinear systems have nonuniform properties, i.e. the L_2-gain of a nonlinear system changes with the operating point (this remark has already be made by several authors, see for instance [4,6]). Other approaches have been proposed to extend the usual definition of L_2-gain to the nonlinear framework, such as the integral input to state stability [10] or the global L_2-gain [6]) (with a state-dependent weight of the control input). Our approach is different in the sense that we do not consider a nonlinear gain function γ of the L_2 norm of the input w (see [11]):

$$\int_0^t \gamma(|w(\tau)|_2) - \alpha(|y(\tau)|_2)\, d\tau \;\geq\; -\beta(|y(0)|_2),\ \forall t \geq 0$$

⋆ This paper was presented at IFAC Symposium on Nonlinear Control (NOLCOS, Enschede, the Netherlands, July 1998).
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(where α, β are $\mathcal{K}_\infty$ functions), but a gain μ function of the state:

$$\tfrac{1}{2}\int_0^t \mu(x(\tau))^2|w(\tau)|^2 - |y(\tau)|^2\, d\tau \geq V(x(t)) - V(x(0))$$

(where V is a storage function). Typically, μ cannot be bounded on the whole state space and a finite L_2-gain cannot be guaranteed in the usual sense. Nevertheless, one may require that the function μ is bounded from above by a certain constant $\gamma > 0$ when the state of the system evolves outside a neighborhood of the origin. If this neighborhood can be made arbitrarily small, adjusting the function μ, the system is said to have *practical* L_2-gain γ. Starting from these new concepts, the *practical* L_2 disturbance attenuation problem (with internal stability) is investigated and it is shown how the usual sufficient conditions for L_2 disturbance attenuation are to be weakened for the "practical" case. Of course, the price to pay is that the more the neighborhood is shrinked, the higher the feedback gain. There is here a compromise to be sought between the effort necessary to *keep the state* in a neighborhood of the origin and the size of this neighborhood.
The paper is organized as follows. In Section 2 definitions and basic properties are given, whereas in Section 3 a collection of examples illustrate such definitions. In Section 4 the interest is focused on the one dimensional case, for which necessary and sufficient conditions are provided to be checked on the data. Then, sections 5 and 6 are devoted to the attenuation problem, with a particular emphasis on single-input single-output systems.

2 A practical L_2-gain property

Consider an input-output dynamical system:

$$\dot{x} = f(x) + k(x)w, \quad y = h(x) \tag{1}$$

where $(x, w, y) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^s$. f, k, h are assumed to be smooth and such that $f(0) = 0$ and $h(0) = 0$. We denote $x(t, x_0, w)$ the solution of (1) at time t with $x(0) = x_0$ and input $w(\cdot)$.
In the following, $\mathcal{O}$ is an open set of R^n, μ a positive function defined on $\mathcal{O}$, γ a positive number and $B(0, \eta)$ the closed ball in $\mathbb{R}^n$ of radius η centered at the origin.

Definition 1 *The system (1) has* **generalized** L_2**-gain** μ *on* $\mathcal{O}$ *if there exists a non-negative* storage *function* V*, locally bounded on* $\mathcal{O}$*, such that for any initial condition* $x_0 \in \mathcal{O}$ *and input* $w \in L_2$ *we have:*

$$\begin{gathered} \forall T > 0 \ s.t.\ x(t, x_0, w) \in \mathcal{O},\ \forall t \in (0, T], \\ V(x_0) \geq V(x(T)) - \tfrac{1}{2}\int_0^T \mu(x(\tau))^2|w(\tau)|^2 - |y(\tau)|^2 d\tau. \end{gathered}$$

The system (1) has **generalized** L_2**-gain** μ **on** $\mathcal{O}$ **with stability** *if furthermore the set* $\mathbb{R}^n \setminus \mathcal{O}$ *is positively invariant and asymptotically stable* [1] *in the absence of input (i.e. for* $w(t) = 0$, $\forall t$*).*

Unlike the linear case, the gain $\mu(\cdot)$ is here sought as a function of the state and not just a constant. If the system (1) has a generalized L_2-gain on $\mathcal{O}$ for a certain μ bounded above by γ, we shall say that it has L_2**-gain** γ **on** $\mathcal{O}$. Then, the usual L_2-gain property is exactly recovered with $\mathcal{O} = \mathbb{R}^n$.
The concepts introduced so far have small practical relevance, as the region $\mathcal{O}$ is yet arbitrary. We focus now on attenuation on particular regions:

Definition 2 *The system (1) has* η-L_2-gain γ *if it has* L_2*-gain* γ *on* $\mathbb{R}^n \setminus B(0, \eta)$*. (1) is said to have* **practical** L_2**-gain** γ *if it has* η-L_2*-gain* γ *for any* $\eta > 0$*.*

Notice that the practical L_2-gain *with stability* property implies the attractivity of the dynamics $\dot{x} = f(x)$: suppose this in not true, then there exists $\bar{x} \neq 0$ such that $x(\bar{x}, t, 0) \notin B(0, \epsilon)$ for any $t \geq 0$ and for a certain $\epsilon > 0$. Then, choosing $\eta \in (0, \epsilon)$ there is a contradiction with the property of η-L_2-gain with stability for (1).

[1] See [14] for a definition of stability w.r.t. a set.

The "practical" property highlighted in Definition 2 is weaker than the L_2-gain property on $\mathbb{R}^n$, as the storage function might depend on η and is not necessarily bounded when x tends towards 0 (see Example 10 in the next section).
Although the notion of L_2-gain is defined for non-differentiable functions V, only C^1 conditions shall be studied in the following. We define the Hamiltonian:

$$\mathcal{H}(V,\mu)(x) = L_f V(x) + \frac{[L_k V(x)]^2}{2\mu(x)^2} + \frac{|h(x)|^2}{2}.$$

Theorem 3 *If there exists a C^1 non-negative function V defined on $\mathcal{O}$, such that:*

$$\mathcal{H}(V,\mu)(x) \leq 0 \qquad \forall x \in \mathcal{O}, \tag{2}$$

then the system (1) has generalized L_2-gain μ on $\mathcal{O}$. If furthermore (2) is a strict inequality and V is a positive definite and proper, then the smallest level set that contains $\mathbb{R}^n \setminus \mathcal{O}$ (if it exists) is invariant and asymptotically stable under the dynamics $\dot{x} = f(x)$.

PROOF. Completing the square in the expression of $\dot{V}$, one has:

$$\begin{aligned} L_f V(x) + L_k V(x) w = {} & L_f V(x) - \frac{\mu(x)^2}{2}\left[w - \frac{L_k V(x)}{\mu(x)^2}\right]^2 \\ & + \frac{[L_k V(x)]^2}{2\mu(x)^2} + \frac{\mu(x)^2}{2}|w|^2 \leq \tfrac{1}{2}\left(\mu(x)^2|w|^2 - |h(x)|^2\right). \end{aligned}$$

Let $w \in L_2$, $x_0 \in \mathcal{O}$ and integrate between 0 and T such that $x(t, x_0, w)$ exists and belongs to $\mathcal{O}$ for any $t \in (0, T]$:

$$\begin{aligned} V(x(T)) - V(x_0) &= \int_0^T L_f V(x(\tau)) + L_k V(x(\tau)) w(\tau) d\tau \\ &\leq \tfrac{1}{2} \int_0^T \mu(x(\tau))^2 |w(\tau)|^2 - |h(x(\tau))|^2 d\tau. \end{aligned}$$

If inequality (2) is strict, V is a Lyapunov function for $\dot{x} = f(x)$ and by LaSalle invariance principle, the smallest V-level set containing $\{x \mid L_f V(x) = 0\}$ is invariant and asymptotically stable. □

To check the practical L_2 gain property of a particular system, a convenient way consists in choosing a monotonic family of functions V_m and use the following corollary (see Example 10 in the next section):

Corollary 4 *If $\{V_m\}_{m \in M}$ is a non increasing family of definite positive and proper C^1 functions defined on $\mathbb{R}^n$ such that:*

i) $\bigcap_{m \in M} \{x \mid V_m(x) \leq 0\} = \{0\}$
ii) $V_m(x) > 0 \Rightarrow \mathcal{H}(V_m, \gamma)(x) < 0, \ \forall m \in M,$

then the system (1) has practical L_2-gain γ with stability.

It is known that any linear stable system (with zero input) has a finite L_2-gain, which is generally not true for nonlinear systems. Nevertheless, we have a nonlinear converse result thanks to the notion of generalized L_2-gain:

Proposition 5 *Assume that the origin is a globally asymptotically stable equilibrium of $\dot{x} = f(x)$ and that there exists a differentiable Lyapunov function V. If the output $y = h(x)$ is such that the set $\mathcal{O} = \{\, x \mid L_f V(x) < -|h(x)|^2/2 \,\}$ is not empty, then the system (1) has a generalized L_2-gain on $\mathcal{O}$.*

PROOF. Take any positive function μ such that

$$\mu(x) \geq \frac{|L_k V(x)|}{\sqrt{-2L_f V(x) - |h(x)|^2}}, \quad \forall x \in \mathcal{O}.$$

Then, we have $\mathcal{H}(V, \mu)(x) \leq 0$, $\forall x \in \mathcal{O}$. $\square$

3 Some illustrative examples

Example 6 $\quad \dot{x} = -x - x^3 + w, \quad y = x.$

This system has L_2-gain $\gamma = 1$, as condition (2) holds on $\mathbb{R}$ with $V(x) = x^2/2$. However, the system has (as well) generalized gain $\mu(x) = 1/\sqrt{1+x^2}$ over $\mathbb{R}$, as $\mathcal{H}\left(x^2/2, 1/\sqrt{1+x^2}\right) = -x^4/2 \leq 0$. This reflects that for large x, the nonlinear term $-x^3$ is sufficiently strong to compensate any L_2 input.

Example 7 $\quad \dot{x} = -\dfrac{x}{1+x^2} + w, \quad y = x.$

Simple calculations show that $\mathcal{H}\left(x^2/2 + x^4/4, 1 + x^2\right) = 0$, i.e. the system does not have a finite L_2-gain over any unbounded subset of $\mathbb{R}$.

Example 8 $\quad \dot{x} = -x\left(\dfrac{\alpha + x^2}{1+x^2}\right) + w, \quad y = x \quad (\alpha \gg 1)$

Simple algebra shows that

$$\mathcal{H}\left(\frac{x^2}{2} - \frac{(\alpha-1)}{2} ln\left(\frac{\alpha + x^2}{\alpha}\right), \frac{1+x^2}{\alpha + x^2}\right) = 0.$$

Hence, the system has L_2-gain $\gamma = 1$, but its gain about the origin, i.e. of the linear approximation, is $1/\alpha \ll 1$.

Example 9 $\quad \begin{cases} \dot{x}_1 = x_2^3 + w \\ \dot{x}_2 = -x_1 - x_2 - x_2^3 \end{cases} \qquad y = x_1.$

Let $\mathcal{O} = \{(x_1, x_2) \,|\, \mathcal{E}(x_1, x_2) > 0\}$ with $\mathcal{E}(x_1, x_2) = x_2^4 + x_2(x_1 + x_2) + (x_1 + x_2)^2 - x_1^2/2$. For any $(x_1, x_2) \in \mathcal{O}$, we have

$$\mathcal{H}\left(\frac{(x_1 + x_2)^2}{2} + \frac{x_2^2}{2}, \max\left(\frac{|x_1 + x_2|}{\sqrt{2\mathcal{E}(x_1, x_2)}}, \epsilon\right)\right) = 0,$$

where $\epsilon > 0$. The shape of $\mathcal{O}$ illustrates the ability of the system to attenuate disturbances when the state evolves in the upper right or lower left quadrants: $x_1 x_2 \geq 0$ but not when it gets close to the line $x_1 + x_2 = 0$ near 0.

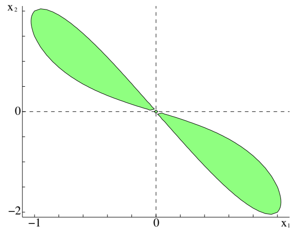

Fig. 1. The set where $\mathcal{E}(x_1, x_2) \leq 0$

Example 10 $\quad \dot{x} = -x^3 + x^2 w, \quad y = x.$

For any $\eta > 0$, $V(x) = \ln(|x|/\eta)$ fulfills (2) on $\mathbb{R} \setminus B(0,\eta)$ with $\mu(x) = 1$. So, the system has practical *L_2-gain $\gamma = 1$ but has not the L_2-gain property in the usual sense, in particular it does not map L_2 inputs into L_2 outputs. To see this, consider the L_2 input: $w(t) = 1$ if $t \in [0,1]$, 0 if $t > 1$, and choose an initial condition $x_0 \in (0,1)$. The state $x(\cdot)$ is non decreasing on the interval $[0,1]$ and decreases asymptotically towards 0 over $(1,+\infty)$. If the system has L_2-gain γ in the classical sense (i.e. in all $\mathbb{R}^n$), there should exists a locally bounded function V such that:*

$$V(x_0) \geq V(x(T)) + \tfrac{1}{2}\int_0^T y(\tau)^2 - \gamma^2 w(\tau)^2 d\tau \geq V(x(T))$$
$$-\tfrac{1}{2}\int_{x(1)}^{x(T)} \frac{dx}{x} - \gamma^2 \geq V(x(T)) + \tfrac{1}{2}\left(\ln\frac{x(1)}{x(T)} - \gamma^2\right)$$

for any $T > 1$. When $T \to \infty$, $x(T) \to 0$, then $V(x_0)$ is unbounded, hence a contradiction.

Example 11 $\quad \dot{x} = -x^3 + w, \quad y = x.$

Consider an initial condition $x_0 \neq 0$ and a L_2 input: $w(t) = x_0^3$ if $t \in [0, x_0^{-6}]$, 0 if $t > x_0^{-6}$. Suppose that the system has L_2-gain γ on $\mathbb{R} \setminus B(0,x_0)$, then (2) yields:

$$x(T) = x_0 \Rightarrow 0 \geq \tfrac{1}{2}\int_0^{x_0^{-6}} y(\tau)^2 - \gamma^2 w(\tau) d\tau = \frac{x_0^{-4} - \gamma^2}{2}.$$

Hence, $|x_0| \geq \gamma^{-1/2}$ and so the system cannot have practical *L_2-gain for any value of γ.*

4 The one dimensional case

Proposition 12 *If the system (1) with $n = 1$ has generalized L_2-gain μ on $\mathcal{O}$, then $\Gamma(x) = \mu(x)^2 f(x)^2 - |k(x)|^2 h(x)^2 \geq 0$, $\forall x \in \mathcal{O}$.*

PROOF. Let V be a storage function. According to [7], its lower semi-continuous envelope V_* is also a storage function. So without any loss of generality, we can assume that V is lower semi-continuous. Recall then a result from [3, Corollary I.17] that states that the set:

$$\mathcal{D} = \{x \in \mathcal{O} \mid \exists \psi \in C^1(\mathcal{O}),\ \psi(x) = V(x), \psi \leq V \text{ near } x\}$$

is dense in $\mathcal{O}$. Take $\bar{x} \in \mathcal{D}$, a neighborhood $\mathcal{N}(\bar{x}) \subset \mathcal{O}$ of $\bar{x}$ and $\psi \in C^1(\mathcal{O})$ such that $\psi(\bar{x}) = V(\bar{x})$ and $\psi \leq V$ on $\mathcal{N}(\bar{x})$. Then, for any $w \in L_2$, there exists $T > 0$ sufficiently small such that:

$$\forall t \in (0,T],\ x(t,\bar{x},w) \in \mathcal{N}(\bar{x}) \ \text{ and}$$
$$\psi(x(t,\bar{x},w)) - \psi(\bar{x}) \leq \tfrac{1}{2}\int_0^t \mu(x(\tau))^2 |w(\tau)|^2 - |y(\tau)|^2 d\tau.$$

Differentiate this last inequality w.r.t. t and write the completion of square:

$$\tfrac{1}{2}\left(\mu(x)w - \frac{\psi_x(x)k(x)}{\mu(x)}\right)^2 - \frac{[\psi_x(x)k(x)]^2}{2\mu(x)^2}$$
$$-\psi_x(x)f(x) - \frac{h(x)^2}{2} \geq 0.$$

This last inequality should be fulfilled for any $w \in \mathbb{R}^q$. It follows that $\Psi = \psi_x(x)$ should satisfy the following inequality

$$\Psi^2 \frac{|k(x)|^2}{2\mu(x)^2} + \Psi f(x) + \frac{h(x)^2}{2} \leq 0, \quad \forall x \in \mathcal{N}(\bar{x})$$

and a necessary condition to have a solution is exactly: $\Gamma(x) \geq 0$, $\forall x \in \mathcal{N}(\bar{x})$. Then the density of $\mathcal{D}$ imposes this condition to be fulfilled for all $x \in \mathcal{O}$. $\square$

Let now re-discuss briefly Examples 7 and 11. In Example 7, a generalized L_2-gain must satisfy $\mu(x) \geq (1 + x^2)$, hence the system does not have L_2-gain on any unbounded set of $\mathbb{R}$. In Example 11, $\Gamma(x) = \mu(x)^2 x^6 - x^2$ shows that we cannot expect a better generalized L_2-gain than $\mu(x) = 1/x^2$.
Finally, we propose sufficient conditions for generalized L_2-gain, that can be checked on the data:

Proposition 13 *Assume that $\Gamma(x) \geq 0$, $\forall x \in \mathcal{O}$ and that the quantity:*

$$Z(x) = -\frac{f(x)\mu(x)^2}{|k(x)|^2}, \quad x \in \mathcal{O}.$$

is well defined (i.e. $|Z(x)| < \infty$, $\forall x \in \mathcal{O}$).
If there exists a subset I of $\mathbb{R}$ such that $I \cap \mathcal{O} \neq \emptyset$ and
1. Z is bounded below on $I \cap \mathcal{O}$,
2. $\forall x_0 \in I \cap \mathcal{O}, \forall x \notin I \cap \mathcal{O},\ f(x)(x - x_0) \leq 0$,
then the system (1) has generalized L_2-gain μ on $\mathcal{O}$.

PROOF. Consider $C_0 = \inf\{Z(\xi) \mid \xi \in I \cap \mathcal{O}\} > -\infty$. Then, take any $x_0 \in I \cap \mathcal{O}$ and write:

$$V(x) = \int_{x_0}^{x} Z(\xi) d\xi - \min(C_0, 0), \quad x \in \mathcal{O}.$$

V is well defined on $\mathcal{O}$ and so
i) $x \in I \cap \mathcal{O} \Rightarrow V(x) \geq C_0 - \min(C_0, 0) \geq 0$,
ii) $x \notin I \cap \mathcal{O} \Rightarrow V(x) \geq -\min(C_0, 0) \geq 0$,
then V is non-negative. Furthermore, one has:

$$\mathcal{H}(V, \mu)(x) = -\frac{\Gamma(x)}{2|k(x)|^2} \leq 0, \quad x \in \mathcal{O}. \qquad \square$$

5 The practical L_2 disturbance attenuation problem

We revisit now the well known problem of L_2 disturbance attenuation [5,12], but for the generalized L_2-gain. For nonlinear systems for which the usual L_2 disturbance attenuation problem might be not solvable, we propose a weaker concept of attenuation, that we call "practical attenuation", which in a sense takes better into account the nonlinear nature of the dynamics. We consider a controlled version of the system (1), i.e.

$$\dot{x} = f(x) + g(x)u + k(x)w, \quad y = h(x) \tag{3}$$

where $u \in \mathbb{R}^p$ is the control and w a disturbance. The problem is: given $\gamma > 0$, find a continuous feedback $u(x)$ such that the closed loop system (3) has practical L_2-gain γ with stability. The considered Hamiltonian is then: $H(V, \gamma, u)(x) = \mathcal{H}(V, \gamma)(x) + L_g V(x) u$.

Proposition 14 *Let V be a C^1 function bounded below on $\mathbb{R}^n$. For any $\epsilon > 0$, consider the set $\mathcal{O}_\epsilon = \{x \in \mathbb{R}^n \mid [L_g V(x)]^2 > \epsilon$ or $\mathcal{H}(V, \gamma)(x) < 0\}$. If $\mathcal{O}_\epsilon \neq \emptyset$, there exists a continuous feedback u_ϵ such that the closed-loop* *system (3) has L_2 disturbance attenuation γ on the set $\mathcal{O}_\epsilon$. Moreover, in the absence of any disturbance, u_ϵ drives the state asymptotically towards the smallest V-level set containing $\mathbb{R}^n \setminus \mathcal{O}_\epsilon$.*

PROOF. Consider the feedback law:

$$u_\epsilon(x) = \left| \begin{array}{l} 0 \ \text{ if } L_gV(x) = 0 \\ \\ \phi(\mathcal{H}(V,\gamma)(x), [L_gV(x)]^2)[L_gV(x)]^T \\ \quad \text{if } \mathcal{H}(V,\gamma)(x) < 0 \text{ or } [L_gV(x)]^2 > \epsilon \\ \phi(\mathcal{H}(V,\gamma)(x), \epsilon)[L_gV(x)]^T \ \text{ otherwise} \end{array} \right.$$

with $\phi(a,b) = -(a + \sqrt{a^2+b^2})/b$, which is continuous on the set $\{x \,|\, \mathcal{H}(V,\gamma)(x) < 0\}$ by Artstein-Sontag's argumentation [9], and is prolongated by continuity on its complementary. Then, for all $x \in \mathcal{O}_\epsilon$, one has:

$$H(V,\gamma,u_\epsilon) = -\sqrt{\mathcal{H}(V,\gamma)(x)^2 + \max(L_gV(x)^4, \epsilon^2)} < 0$$

and the claim is established by Theorem 3. □

An application of the theory discussed in this section can be found in [1].

6 Practical L_2 disturbance attenuation for SISO systems

Consider single-input single-output (SISO) systems in normal form:

$$\begin{cases} \dot{z} &= f_0(z,\xi_1) + k_0(z,\xi_1)w \\ \dot{\xi}_1 &= \xi_2 + k_1(z,\xi_1)w \\ \dot{\xi}_2 &= \xi_3 + k_2(z,\xi_1,\xi_2)w \\ \vdots & \\ \dot{\xi}_{r-1} &= \xi_r + k_{r-1}(z,\xi_1,\ldots,\xi_{r-1})w \\ \dot{\xi}_r &= u + k_r(z,\xi_1,\ldots,\xi_r)w \\ y &= h_0(z,\xi_1) \end{cases} \tag{4}$$

f_0, h_0, k_i are smooth and such that $f_0(0,0) = 0$ and $h_0(0,0) = 0$. Geometric conditions for a general system (3) to be (globally) diffeomorphic to a system in normal form (4) are well known, see [5].
Define the Hamiltonian function $\mathcal{H}_0(V,\gamma)$ to be:

$$\frac{\partial V}{\partial z}(z) f_0(z,0) + \frac{1}{2\gamma^2}\left[\frac{\partial V}{\partial z}(z) k_0(z,0)\right]^2 + \frac{|h_0(z,0)|^2}{2}.$$

Lemma 15 *Consider a system in the form (4) with $r = 1$. If for any $\eta > 0$, there exists a C^1 proper function V on $\mathbb{R}^{n-1}$ such that $S_V = \{z \,|\, \mathcal{H}_0(V,\gamma)(z) \geq 0 \text{ or } V(z) < 0\} \subset B(0,\eta)$, then there exists a C^1 feedback u, with $u(0) = 0$, such that the closed loop system has practical L_2 disturbance attenuation level γ with stability.*

PROOF. Write $f_0(z,\xi) = f_0(z,0) + \bar{f}_0(z,\xi)\xi$, $k_0(z,\xi) = k_0(z,0) + \bar{k}_0(z,\xi)\xi$, $h_0(z,\xi) = h_0(z,0) + \bar{h}_0(z,\xi)\xi$, and consider the candidate storage function $W(z,\xi) = V(z) + \xi^2/2$. Then, there exists a smooth function $b : (z,\xi) \longmapsto b(z,\xi)$ such that: $H(W,\gamma,u)(z,\xi) = \mathcal{H}_0(V,\gamma)(z) + [u + b(z,\xi)]\xi$. Sufficient requirements for u are:
1. Compensate the term $b(z,\xi)$ if $|z| \geq \eta$.

2. Guarantee $H(W,\gamma,u)(z,\xi)<0$ if $|z|<\eta$ and $|\xi|\geq\eta$.
3. Be C^1 and such that $u(0)=0$.
As S_V is bounded, define $M=\max_{z\in S_V}\mathcal{H}_0(V,\gamma)(z)$ and consider the C^1 function from $\mathbb{R}$ to $[0,1]$:

$$\psi(\lambda)=\left|\begin{array}{l} 0 \quad \text{if } \lambda<\eta^2/4 \\ 1 \quad \text{if } \lambda>\eta^2 \\ 2\left(\frac{\eta^2-\lambda}{3/4\eta^2}\right)^3-3\left(\frac{\eta^2-\lambda}{3/4\eta^2}\right)^2+1 \quad \text{otherwise}\end{array}\right.$$

Then the feedback law:

$$u(z,\xi)=-\psi(z^2+\xi^2)\left[\frac{M}{\eta}+b(z,\xi)\right] \tag{5}$$

guarantees that $H(W,\gamma,u)(z,\xi)<0$ for all $(z,\xi)\notin S_V\times B(0,\eta)$. Proposition 14 gives then the conclusion. □

Lemma 16 *Consider a system in the form (4) with $r=1$. Assume that for any $\eta>0$ there exists a C^1 proper function V on $\mathbb{R}^{n-1}$ and a smooth feedback $v(z)$ such that:*

$$\begin{aligned}\bar{S}_V=\{\,z\,|\,&\frac{\partial V}{\partial z}(z)f_0(z,v(z))+\frac{1}{2\gamma^2}\left[\frac{\partial V}{\partial z}(z)k_0(z,v(z))\right]^2\\ &+\frac{|h_0(z,v(z))|^2}{2}\geq 0 \text{ or } V(z)<0\,\}\subset B(0,\eta)\end{aligned}$$

then the system (4) has practical L_2 disturbance attenuation level γ with stability.

PROOF. Make the change of variable $\theta=\xi-v(z)$, then

$$\left\{\begin{aligned}\dot{z}&=f_0(z,v(z)+\theta)+k_0(z,v(z)+\theta)w\\ &=\bar{f}(z,\theta)+\bar{k}_0(z,\theta)w\\ \dot{\theta}&=u+k_1(z,v(z)+\theta)w-\frac{\partial v}{\partial z}(z)f_0(z,v(z)+\theta)\\ &=\bar{u}+\bar{k}_1(z,\theta)w\end{aligned}\right.$$

with $\bar{u}=u-\partial v/\partial z\ f_0(z,v(z)+\theta)$. Lemma 15 can then be applied to the system (z,θ) with $S_V=\bar{S}_V$, which provides the existence of a smooth feedback $\bar{u}$ such that the system has L_2 disturbance attenuation outside the set $S_W=\bigcup_{z\in\bar{S}_V}\{z\}\times B(v(z),\eta)$. Finally, by formula (5), $S_W\subset B(0,2\eta)$, and by Proposition 14, the claimed result is proved. □

Now, a recursive application of this lemma in a backstepping-like procedure [8] gives the result:

Theorem 17 *If for any $\eta>0$, there exists a C^1 proper function V on $\mathbb{R}^{n-r}$ such that $\{z\,|\,\mathcal{H}_0(V,\gamma)(z)\geq 0$ or $V(z)<0\}\subset B(0,\eta)$, then the system (4) has practical L_2 disturbance attenuation γ with stability.*

PROOF. It is obtained applying first Lemma 15 on the sub-system (z,ξ_1) which provides a C^1 feedback u_1. Lemma 16 can then be applied recursively. □

Corollary 18 *Suppose $r=n-1$. Suppose moreover that the following holds.*
i) $\gamma^2 f_0(z,0)^2-|k_0(z,0)|^2h_0(z,0)^2\geq 0,\quad \forall z\neq 0.$
ii) $|f_0(z,0)|/|k_0(z,0)|^2<\infty,\quad \forall z\neq 0,$
iii) There exists a bounded interval I of $\mathbb{R}$, not reduced to $\{0\}$, such that, $\forall z_0\in I,\ \forall z\notin I,\ f_0(z,0)(z-z_0)\leq 0$.
Then the practical L_2 disturbance attenuation problem with stability is solvable for the system (4).

PROOF. The function $z \mapsto |f_0(z,0)|/|k_0(z,0)|^2$ is continuous on $\mathbb{R} \setminus \{0\}$ so bounded on $\mathbb{R} \setminus B(0,\eta)$ for any $\eta > 0$. For small enough η, $I \cap \mathbb{R} \setminus B(0,\eta)$ is not empty and one can apply Proposition 13. $\square$

Example 19

$$\begin{cases} \dot{z} &= -z^3 + z^2\xi_1 + z^2 w, \qquad y = z \\ \dot{\xi}_1 &= \xi_2 + k_1(z,\xi_1)w \\ \vdots & \\ \dot{\xi}_{r-1} &= \xi_r + k_{r-1}(z,\xi_1,\ldots,\xi_{r-1})w \\ \dot{\xi}_r &= u + k_r(z,\xi_1,\ldots,\xi_r)w \end{cases}$$

For any $\eta > 0$, take the function $V(z) = ln(|z|/\eta)$ (for $z \neq 0$). Then, $\mathcal{H}_0(V,1)(z) = 0$, $\forall z \neq 0$ and $V(z) < 0$ for $|z| < \eta$ (see Example 10). So, according to Theorem 17, we conclude that such systems have practical L_2 disturbance attenuation (with $\gamma = 1$) with stability. Notice that we cannot apply the results of [6] because the linearized system at the origin is not controllable.

7 Conclusion

In this paper, we have presented an extension of the concept of L_2-gain for nonlinear systems. Having in mind that a nonlinear system can possess locally (but not globally) a finite L_2-gain in the usual sense, we consider generalized L_2-gain as a function of the state. This new definition leads us to revisit the L_2 disturbance attenuation problem for nonlinear systems.

The proposed definition of practical L_2-gain stays to the notion of L_2-gain as the notion of practical stability is related to the concept of asymptotic stability. The remarkable point is that for some well known nonlinear systems (see [1]), although the strict attenuation of L_2 disturbances is not possible, it can be practically achieved.

One of the advantages of this approach is to provide characterizations in terms of solutions of Hamilton-Jacobi inequalities. So, similarly to the nonlinear H_∞ problem (where the best guaranteed L_2-gain γ^* is sought among constants), we expect that numerical schemes, such as the ones based on viscosity solutions (see for instance [2]), can be adapted to this generalized L_2-gain framework.

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