Chapter 1

H_∞-norm approximation

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1.1 Description of the problem

Let $\mathcal{RH}_\infty^m$ be the (Hardy) space of real-rational scalar[1] transfer functions of order m, bounded on the imaginary axisand analytic into the right-half complex plane.The optimal approximation problem in the H_∞ norm can be statedas follows.

(A*) (Optimal Approximation in the H_∞ norm)
Given $G(s) \in \mathcal{RH}_\infty^N$ and an integer $n < N$ find[2], if possible, $A^\star(s) \in \mathcal{RH}_\infty^n$ such that

$$A^\star(s) = \arg\min_{A(s)\in\mathcal{RH}_\infty^n} \|G(s) - A(s)\|_\infty. \tag{1.1}$$

[1]Similar considerations can be done for the non-scalar case.
[2]By *find* we mean *find an exact solution* or an algorithm converging to the exact solution.

For such a problem, let

$$\gamma_n^\star = \min_{A(s)\in\mathcal{RH}_\infty^n} \|G(s) - A(s)\|_\infty,$$

then two further problems can be posed.

(D) (Optimal Distance problem in the H_∞ norm)
Given $G(s) \in \mathcal{RH}_\infty^N$ and an integer $n < N$ find $\gamma_n^\star$.

(A) (Sub-optimal Approximation in the H_∞ norm)
Given $G(s) \in \mathcal{RH}_\infty^N$, an integer $n < N$ and $\gamma > \gamma_n^\star$ find $\tilde{A}(s) \in \mathcal{RH}_\infty^n$ such that

$$\gamma_n^\star \le \|G(s) - \tilde{A}(s)\|_\infty \le \gamma.$$

The optimal H_∞ approximationproblem can be formally posed as a constrained min-max problem. For, note that any function in $\mathcal{RH}_\infty^n$ can be putin a one to one correspondence with a point θ of some (open) set $\Omega \subset R^{2n}$, therefore the problem of computing $\gamma_n^\star$can be posed as

$$\gamma_n^\star = \min_{\theta\in\Omega}\max_{\omega\in R} \|G(j\omega) - A(j\omega)\|, \tag{1.2}$$

where $A(s) = A(s,\theta)$. The above formulation provides a brute force approach to the solution of theproblem. Unfortunately, this method is not of any use in general, because of the complexity of the set Ω and because of the curse of dimensionality. However, the formulation (1.2) suggests that possible candidate solutions of the optimal approximation problem are the saddle points of the function

$$\|G(j\omega) - A(j\omega,\theta)\|,$$

which can be, in principle, computed using numerical tools. It would be interesting to prove (or disprove) that

$$\min_{\theta\in\Omega}\max_{\omega\in R} \|G(j\omega) - A(j\omega,\theta)\| = \max_{\omega\in R}\min_{\theta\in\Omega} \|G(j\omega) - A(j\omega,\theta)\|.$$

The solution method based on the computation of saddle points does not give any insight into the problem, neither exposes any systems theoretic interpretation of the optimal approximant. An interesting property of the optimal approximant is stated in the following simple fact, which can be used to rule out that a candidate approximant is optimal.

Fact. Let $A^\star(s) \in \mathcal{RH}_\infty^n$ be such that equation (1.1) holds. Suppose

$$|W(j\omega^\star) - A^\star(j\omega^\star)| = \gamma_n^\star, \tag{1.3}$$

and

$$A(j\omega^\star) \ne 0 \tag{1.4}$$

for $\omega^\star = 0$. Then there exists a constant $\tilde{\omega} \ne \omega^\star$ such that

$$|W(j\tilde{\omega}) - A^\star(j\tilde{\omega})| = \gamma_n^\star,$$

i.e. if the value $\gamma_n^\star$ is attained by the function $|W(j\omega) - A^\star(j\omega)|$at $\omega = 0$ it is also attained at some $\omega \neq 0$.

Proof. We prove the statement by contradiction. Suppose

$$|W(j\omega) - A^\star(j\omega)| < \gamma_n^\star, \tag{1.5}$$

for all $\omega \neq \omega^\star$ and consider the approximant $\tilde{A}(s) = (1+\lambda)A^\star(s)$, with $\lambda \in \mathbb{R}$. By equation (1.5), condition (1.4) and by continuity with respect to λ and ω of

$$|W(j\omega) - \tilde{A}(j\omega)|,$$

there is a $\lambda^\star$ (sufficiently small) such that

$$\max_{\omega} |W(j\omega) - (1+\lambda^\star)A^\star(j\omega)| < \gamma_n^\star,$$

or, what is the same, it is possible to obtain an approximant which is better then $A^\star(s)$, hence a contradiction. $\lhd$

It would be interesting to show that the above fact holds (or it does not hold) when $\omega^\star \neq 0$.

1.2 Available results and possible solution paths

Approximation and model reduction have always been central issues in system theory. For a recent survey on model reduction in the large-scale setting we refer the reader to the book [1]. There are several results in this area. If the approximation is performed in the Hankel norm then an explicit solution of the optimal approximation and model reduction problems has been given in [3]. Note that this procedure provides, as a byproduct, an upper bound for $\gamma_n^\star$ and a solution of the sub-optimal approximation problem. If the approximation is performed in the H_2 norm, several results and numerical algorithms are available[4]. For approximation in the H_∞ norm a conceptual solution is given in [5]. Therein it is shown that the H_∞ approximation problem can be reduced to a Hankel norm approximation problem for an extended system (*i.e.* a system obtained from a state space realization of the original transfer function $G(s)$ by adding inputs and outputs). The extended system has to be constructed with the constraint that the corresponding Grammians P and Q satisfy

$$\lambda_{\min}(PQ) = (\gamma_n^\star)^2 \quad \text{with multiplicity} \quad N - n. \tag{1.6}$$

However, the above procedure, as also noted by the authors of [5], is not computationally viable, and presupposes the knowledge of $\gamma_n^\star$. Hence the need for further study of the problem. In the recent paper [2] the decay rates of the Hankel singular values of stable, single-input single-output systems, are studied. Let $G(s) = \frac{p(s)}{q(s)}$ be the transfer function under consideration. The decay rate of the Hankel singular values is studied by introducing a new set of input/output system invariants, namely the quantities $\frac{p(s)}{q^*(s)}$, where $q(s)^* = q(-s)$, evaluated at the poles of $G(s)$. These results are expected to yield light into the structure of the above problem (1.6). Another paper of interest especially for the suboptimal approximation case, is [6]. In this paper the set of all systems whose H_∞ norm is less than some positive number γ is parameterized. Thus the following problem can be posed: given such a system with H_∞ norm less than γ, find conditions under which it can be decomposed in the sum of two systems, one of which is prespecified. Finally, there are two special classes of systems which may be studied to improve our insight into the general problem. The first class is composed of single-input single-output discrete-time stable systems. For such systems, an interesting related problem is the *Carathéodory-Fejér (CF)* approximation problem which is used for elliptic filters design. In [7] it is shown that in the scalar, discrete-time case, optimal approximants in the Hankel norm approach asymptotically optimal approximants in the H_∞ norm (the asymptotic behavior being with respect to $\epsilon \to 0$, where $|z| \leq \epsilon < 1$). The CF problem through the contribution of Adamjan-Arov-Krein and later Glover, evolved into what is nowadays called the *Hankel-norm approximation* problem. However, no asymptotic results have been shown to hold in the general case. The second special class is that of symmetric systems, that is, systems whose state space representation (C, A, B) satisfies $A = A'$ and $B = C'$. For instance, these systems have a positive definite Hankel operator and have further properties that can be exploited in the construction of approximants in the H_∞ sense.

Bibliography

[1] A.C. Antoulas, *Lectures on the approximation of large scale dynamicalsystems*, SIAM, Philadelphia, to appear (2002).

[2] A.C. Antoulas, D.C. Sorensen, and Y. Zhou, *On the decay rate of Hankel singular values and related issues*, Systems and Control Letters (2002).

[3] K. Glover, *All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds*, International Journal of Control, **39**: 1115-1193 (1984).

[4] X.-X. Huang, W.-Y. Yan and K.L. Teo, *H_2 near optimal model reduction*, IEEE Trans. Automatic Control, **46**: 1279-1285 (2001).

[5] D. Kavranoglu and M. Bettayeb, *Characterization of the solution to the optimal H_∞ model reduction problem*, Systems and Control Letters, **20**: 99-108 (1993).

[6] H.G. Sage and M.F. de Mathelin, *Canonical H_∞ state space parametrization*, Automatica, July 2000.

[7] L.N. Trefethen, *Rational Chebyshev approximation on the unitdisc*, Numerische Mathematik, **37**: 297-320 (1981).