# Implementation of Distributed Delay in Control Laws*

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**Abstract**

This note proposes two approaches to approximate the distributed delay in control laws and, furthermore, to implement it in the $z$-domain and in the $s$-domain. The $H^\infty$-norm of the approximation error converges to 0 when the number $N$ of approximation steps approaches $+\infty$. Hence, the instability problem due to the approximation error (which has been widely studied in recent years) does not exist, provided that $N$ is large enough. Moreover, the static gain is guaranteed so that no extra efforts are needed to retain the steady-state performance. As by-products, two new formulae for the forward and backward rectangular rules are obtained. These formulae are more accurate than the conventional ones when the integrand has an exponential term.

**Index Terms**: distributed delay, finite-spectrum assignment, dead-time compensator, quadrature approximation, numerical integration, implementation error

## 1 Introduction

Distributed delays (i.e., finite integrals over the time, also called finite-impulse-response FIR blocks) often appear as a part of dead-time compensators for processes with dead time, in particular, for unstable processes with dead time [1, 2, 3, 4]. They also appear in $H^\infty$ control of (even, stable) dead-time systems [5, 6, 7, 8, 9] and continuous-time deadbeat control [10]. Due to the requirement of internal stability, such an FIR block has to be, approximately, implemented as a stable block without hidden unstable poles.

A common way to do so is to replace the distributed delay by the sum of a series of discrete (often commensurate) delays [1, 2, 3] (other interesting implementations using resetting mechanism can be found in [11] and [12]). However, it has emerged very recently that this approximation method (more specifically, using quadrature rules such as rectangular, trapezoidal and Simpson's rules etc.) cannot guarantee the system stability, even when quite accurate approximation integral laws were used [13]. This topic has drawn a lot of attention from the delay community and has become a very hot topic in recent years; see [12, 13, 14, 15, 16, 17, 18, 19, 20]. It has been proposed as an open problem in the survey paper [21]. The analysis of the causes of such behaviors was studied in [17, 14, 13] using a simple example. It was shown in [17] that the resulting system becomes a neutral time-delay system and the closed-loop poles having large magnitude located in the right-half plane (whatever the precision of the trapezoidal approximation is) caused the instability.

It has been well understood that the existence of a low-pass filter in the approximation may fix the instability problem, as explicitly or implicitly reported in [19, 20, 18]. Indeed, this is a standard technique to convert a neutral time-delay system into a retarded one; see e.g. [22]. However, it is not clear why the approximation, which only involves the classical quadrature rules and Laplace transform, has lost the inherent low-pass property, nor is it clear how to easily choose a suitable low-pass filter (see [18, Subsection 4.2]).

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This note intends to answer these questions and proposes some improved approximations to implement the distributed delay. In this note, an *approximation* is called an *implementation* only when it is implementable.

In the literature, this problem is often considered in the context of a control system. It often involves the change of the control structure, e.g., due to an algebraic loop, inserting a low-pass filter or even the redesign of a control law. Here, this problem is regarded as a pure approximation/implementation problem in the frequency domain. Two different reasonings will be applied, but the obtained results are the same. The proposed implementations guarantee both the low frequency behavior and the high frequency behavior. Moreover, the $H^\infty$-norm of the approximation error converges to $0$ when the approximation step $N$ approaches $+\infty$. Hence, there is no change of control structure; there is no instability provided that $N$ is large enough. Indeed, a widely studied system, which demonstrated instability, is stable even when $N=1$.

The rest of this note is organized as follows. Some preliminaries are given in Section 2. Two different approaches are proposed to approximate the distributed delay in Section 3 and implementations in the $z$-domain and in the $s$-domain are proposed in Section 4. The stability issue is discussed in Section 5 and numerical examples are given in Section 6.

# 2 Preliminaries

It has been well known [1, 23] that, for a given dead-time process $P(s)e^{-sh}$ with $P(s)=C(sI-A)^{-1}B$, the finite-spectrum-assignment control law is given by

$$u(t)=Fx_p(t), \qquad x_p(t)=e^{Ah}x(t)+\int_0^h e^{A\zeta}Bu(t-\zeta)d\zeta, \tag{1}$$

where $F$ is the state feedback gain and $x_p(t)$ is the predicted state of the process. This control law stabilizes the system to a finite spectrum if $A+BF$ is stable. Denote the distributed delay in (1) to be

$$v(t)=\int_0^h e^{A\zeta}Bu(t-\zeta)d\zeta, \tag{2}$$

then the $s$-domain equivalent (i.e., the transfer function from $u$ to $v$) is:

$$Z(s)=(I-e^{-(sI-A)h})(sI-A)^{-1}B. \tag{3}$$

Hence, in the frequency domain, the *distributed* delay $Z$ is actually a system including a *discrete* delay but with a special property that all poles are canceled by its zeros, i.e., an entire function. This paves the way that the techniques mentioned in [24] can be applied to approximate $Z$ with a rational function, with special attention paid to avoiding the unstable poles.

The integral (distributed delay) $v(t)$ from (2) can be approximated in the time domain by using various quadrature rules such as rectangular, trapezoidal and Simpson's rules etc. In this note, the analysis is based on the rectangular rule for simplicity. It can be easily extended to other rules. The approximated $v(t)$ using the backward rectangular rule is

$$v_w(t)=\frac{h}{N}\sum_{i=1}^{N}e^{iA\frac{h}{N}}Bu(t-i\frac{h}{N}), \tag{4}$$

where $N$ is the number of approximation steps. The Laplace transformation of $v_w(t)$, as done in the literature, gives the following approximation of $Z(s)$ (i.e., the transfer function from $u$ to $v_w$):

$$Z_w(s)=\frac{h}{N}\cdot\sum_{i=1}^{N}e^{-i\frac{h}{N}(sI-A)}B. \tag{5}$$

As is known, $Z_w$ is not a good implementation of $Z$ [13, 14, 18]. A simple reason to question this is that the original FIR block $Z$ is strictly proper but $Z_w$ is not. The bad approximation at high frequencies makes the stability analysis *unnecessarily* complicated and, what is worse, makes no guarantee of the system stability [14, 15, 13, 17, 19].

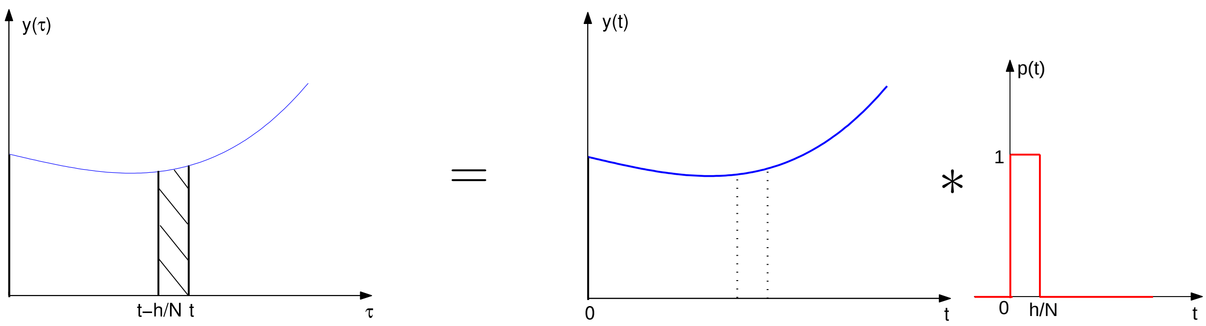



Figure 1: Illustration of Proposition 1

# 3 Approximation of distributed delay

## 3.1 The integration $\int_0^{\frac{h}{N}} y(t-\tau)d\tau$

**Proposition 1.** *For any integrable function* $y(t)$ *and the step function* $1(t)$*, the following identity holds:*

$$\int_0^{\frac{h}{N}} y(t-\tau)d\tau = \int_{t-\frac{h}{N}}^{t} y(\tau)d\tau = y(t) * p(t),$$

*where* $*$ *stands for the convolution and* $p(t) = 1(t) - 1(t - \frac{h}{N})$ *is a rectangular pulse function.*

*Proof.* The first "$=$" is obvious. The second "$=$" can be proved using the definition of the convolution. An illustration of this formula is shown in Figure 1. $\square$

This formula seems trivial, but it reveals the secret behind the instability phenomenon. Obviously, the Laplace transformation of $\int_0^{\frac{h}{N}} y(t-\tau)d\tau$ is $Y(s) \cdot \frac{1-e^{-s\frac{h}{N}}}{s}$, where $Y(s)$ is the Laplace transformation of $y(t)$ and $\frac{1-e^{-s\frac{h}{N}}}{s}$ is that of $p(t)$. However, when a quadrature rule is applied to approximate $\int_0^{\frac{h}{N}} y(t-\tau)d\tau$, then the corresponding Laplace transformation is $Y(s)$ multiplied by a polynomial of delays. For example, when the forward rectangular rule is used, i.e.,

$$\int_0^{\frac{h}{N}} y(t-\tau)d\tau \approx y(t) \cdot \frac{h}{N},$$

this polynomial is $\frac{h}{N}$; when the backward rectangular rule is used, this polynomial is $\frac{h}{N}e^{-\frac{h}{N}s}$; when the trapezoidal rule is used, this polynomial is $\frac{h}{N}\frac{1+e^{-\frac{h}{N}s}}{2}$. Hence, in the frequency domain, the quadrature approximation can be interpreted as approximating $\frac{1-e^{-s\frac{h}{N}}}{s}$ with a polynomial of delay $e^{-\frac{h}{N}s}$. This approximation loses the strict properness of $\frac{1-e^{-s\frac{h}{N}}}{s}$. An approximation, which does not lose this important property, will be given in Subsection 4.2.

## 3.2 Approximation in the $s$-domain via the Laplace transform

Divide the interval $[0,\ h]$ into $N$ sub-intervals $[i\frac{h}{N},\ (i+1)\frac{h}{N}]$, $i = 0,\ 1,\ \ldots,\ N-1$, then $v(t)$ in (2) can be re-written as

$$v(t) = \sum_{i=0}^{N-1} \int_{i\frac{h}{N}}^{(i+1)\frac{h}{N}} e^{A\zeta} B u(t-\zeta)d\zeta. \tag{6}$$

When $N$ is chosen to be large enough, $e^{A\zeta}$ in the interval $[i\frac{h}{N},\ (i+1)\frac{h}{N}]$ can be well approximated by $e^{iA\frac{h}{N}}$. This offers the following approximation for $v(t)$:

$$\begin{aligned} v(t) \approx v_f(t) &= \sum_{i=0}^{N-1} e^{iA\frac{h}{N}} B \int_{i\frac{h}{N}}^{(i+1)\frac{h}{N}} u(t-\zeta)d\zeta \\ &= \sum_{i=0}^{N-1} e^{iA\frac{h}{N}} B \int_{0}^{\frac{h}{N}} u(t-i\frac{h}{N}-\tau_i)d\tau_i, \end{aligned} \tag{7}$$

where the variable changes $\zeta = \tau_i + i\frac{h}{N}$ are used and the subscript "$f$" stands for forward. This approximation can also be obtained by applying the technique used in [20], which involves block-pulse functions. The reasoning used here is simpler and needs less mathematical background. Applying Lemma 1, the last formula becomes

$$v_f(t) = \sum_{i=0}^{N-1} e^{iA\frac{h}{N}} Bu(t-i\frac{h}{N}) * p(t). \tag{8}$$

On the other hand, if it is assumed that

$$u(t-i\frac{h}{N}-\tau_i) = u(t-i\frac{h}{N}) \qquad \text{for} \qquad 0 \le \tau_i < \frac{h}{N}, \tag{9}$$

then $v_f$ given in (7) can be *further* approximated as

$$v_f(t) \approx v_{wf}(t) = \frac{h}{N}\sum_{i=0}^{N-1} e^{iA\frac{h}{N}} Bu(t-i\frac{h}{N}). \tag{10}$$

This is exactly the approximation of $v$ by using the forward rectangular rule. As will be shown later, the approximation $v_f$ does not cause instability when $N$ is large enough. However, as is known, the approximation $v_{wf}$ does. The significant difference between $v_f$ and $v_{wf}$ is the convolution with $p(t)$. An alternative interpretation is that the condition (9) is not explicitly shown in (10).

The transfer function from $u$ to $v_f$, according to (8), gives the following approximation of $Z(s)$:

$$Z_f(s) = \frac{1-e^{-s\frac{h}{N}}}{s} \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B. \tag{11}$$

Similarly, the transfer function from $u$ to $v_{wf}$ is

$$Z_{wf}(s) = \frac{h}{N}\sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B. \tag{12}$$

**Theorem 2.** *The approximation $Z_f$ holds the following properties: (i) $\lim_{N\to+\infty} Z_f(s) = Z(s)$; (ii) $Z_f$ is strictly proper, i.e., $\lim_{|s|\to+\infty,\,\Re(s)\ge 0} Z_f(s) = 0$.*

*Proof.*
$$\begin{aligned} \lim_{N\to+\infty} Z_f(s) &= \lim_{N\to+\infty} \tfrac{1-e^{-s\frac{h}{N}}}{s} \cdot \textstyle\sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B \\ &= \lim_{N\to+\infty} \frac{1-e^{-s\frac{h}{N}}}{s} (I-e^{-(sI-A)h})(I-e^{-\frac{h}{N}(sI-A)})^{-1}B \\ &= (I-e^{-(sI-A)h}) \lim_{N\to+\infty} \frac{1-e^{-s\frac{h}{N}}}{s} (I-e^{-\frac{h}{N}(sI-A)})^{-1}B \\ &= (I-e^{-(sI-A)h}) \lim_{\tau\to 0^+} \frac{1-e^{-s\tau}}{s} (I-e^{-\tau(sI-A)})^{-1}B \\ &= (I-e^{-(sI-A)h}) \lim_{\tau\to 0^+} e^{-s\tau} \cdot ((sI-A)e^{-\tau(sI-A)})^{-1}B \\ &= (I-e^{-(sI-A)h})(sI-A)^{-1}B = Z(s), \end{aligned}$$

where the substitution $\tau = h/N$ is used. The second property is obvious. This completes the proof. □

*Remark* 1. Actually, $\lim_{N\to+\infty} Z_{wf}(s) = Z(s)$ as well. However, $Z_{wf}$ is not strictly proper. This makes the stability analysis of the system, as done in the literature, very complicated. Furthermore, as will be proved later, $Z_f$ converges to $Z$ uniformly.

Although the approximation (11) guarantees the high-frequency behavior of the distributed delay $Z$, the approximation error at low frequencies might be large. In particular, the non-zero error at the zero frequency is not desirable. It changes the system performance at the steady state, as can be seen from the simulations in [13], and hence extra efforts to guarantee the steady-state performance are needed [2]. This means certain change of control law, e.g., as used in [18, Example 2], is needed. Such efforts can be eliminated by using a different approximation as follows.

Instead of approximating $e^{A\zeta}$ with $e^{iA\frac{h}{N}}$ as in (7), it can be approximated with the mean value of $e^{A\zeta}$ in the interval. This offers the following approximation[1]:

$$\begin{aligned} v(t) \approx v_{f0}(t) &= \sum_{i=0}^{N-1} \frac{N}{h} \int_{i\frac{h}{N}}^{(i+1)\frac{h}{N}} e^{A\zeta} d\zeta \cdot B \cdot \int_{i\frac{h}{N}}^{(i+1)\frac{h}{N}} u(t-\zeta) d\zeta \\ &= (e^{\frac{h}{N}A} - I)(\frac{h}{N}A)^{-1} \cdot v_f(t). \end{aligned}$$

The corresponding approximation of $Z$ in the $s$-domain is given by

$$Z_{f0}(s) = \frac{1 - e^{-\frac{h}{N}s}}{s} \frac{e^{\frac{h}{N}A} - I}{\frac{h}{N}} A^{-1} \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B. \tag{13}$$

**Theorem 3.** *$Z_{f0}$ holds the following properties: (i) $\lim_{N\to+\infty} Z_{f0}(s) = Z(s)$; (ii) $Z_{f0}$ is strictly proper; (iii) $\lim_{s\to 0} Z_{f0}(s) = \lim_{s\to 0} Z(s)$.*

*Proof.* Property (i) is obvious since $\lim_{N\to+\infty} \frac{N}{h}(e^{\frac{h}{N}A} - I)A^{-1} = I$ and property (ii) is also obvious.

The static gain of $Z_{f0}$ is the same as that of $Z$ because

$$\begin{aligned} \lim_{s\to 0} Z_{f0}(s) &= \lim_{s\to 0} \frac{1 - e^{-s\frac{h}{N}}}{s} \frac{e^{\frac{h}{N}A} - I}{\frac{h}{N}} A^{-1} \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B \\ &= (e^{\frac{h}{N}A} - I) A^{-1} \sum_{i=0}^{N-1} e^{i\frac{h}{N}A} B \\ &= -(I - e^{Ah}) A^{-1} B = \lim_{s\to 0} Z(s). \end{aligned}$$

□

$Z_{f0}$ guarantees a small approximation error at both low and high frequencies, in particular, zero error at the frequencies $0$ and $+\infty$. Hence, $Z_{f0}$ is more accurate than $Z_f$, in particular, at low frequencies. This indicates that a similar change in the rectangular rule may provide a better accuracy for numerical integration. This better approximation formula is

$$\int_0^h e^{A\zeta} B u(t-\zeta) d\zeta \approx (e^{\frac{h}{N}A} - I) A^{-1} \cdot \sum_{i=0}^{N-1} e^{iA\frac{h}{N}} B u(t - i\frac{h}{N}). \tag{14}$$

To the best knowledge of the author, this result is new. When $A = 0$, $(e^{\frac{h}{N}A} - I)A^{-1}$ becomes $\frac{h}{N}$ and hence this new formula can be regarded as an extension of the conventional forward rectangular rule. It provides a better approximation than the conventional forward rectangular rule when the integrand has an exponential term.

[1]If $A$ is singular, then an appropriate limitation should be used to calculate some elements of $\frac{N}{h}(e^{\frac{h}{N}A} - I)A^{-1}$ when necessary. It can also be replaced by the integral $\frac{N}{h}\int_0^{\frac{h}{N}} e^{A\zeta} d\zeta$. Similar situations are for $(I - e^{Ah})A^{-1}$ and $\frac{N}{h}(I - e^{-\frac{h}{N}A})A^{-1}$.

### 3.3 Direct approximation in the $s$-domain

In the $s$-domain, the distributed delay $Z$ from (3) can be easily[2] re-written as

$$Z(s) = (I - e^{-(sI-A)\frac{h}{N}})(sI-A)^{-1} \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)} B. \tag{15}$$

Hence, the distributed delay $Z$ has been converted to the sum of a series of discrete delays, although there still exist hidden unstable poles in the first part, i.e.,

$$H(A) = (I - e^{-(sI-A)\frac{h}{N}})(sI-A)^{-1}.$$

It is relatively easy to implement $H(A)$ because it involves a much shorter delay $h/N$ when $N$ is large.

The above $H(A)$ is a function of matrix $A$. It can be expanded as the following power series of $A$:

$$H(A) = \frac{1-e^{-\frac{h}{N}s}}{s} I + \frac{1-e^{-\frac{h}{N}s} - \frac{h}{N}se^{-\frac{h}{N}s}}{s^2} \cdot A \; + \frac{1}{2!}\frac{2(1-e^{-\frac{h}{N}s}) - \frac{h}{N}s(2+\frac{h}{N}s)e^{-\frac{h}{N}s}}{s^3} \cdot A^2 + \cdots.$$

This series (uniformly) converges for any square matrix $A$, provided that $H(sI)$ is defined to be $\frac{h}{N}I$ [25]. The approximation by the first term $H(A) \approx \frac{1-e^{-\frac{h}{N}s}}{s} I$ provides the approximation $Z_f(s)$ given in (11). Note that the hold filter $\frac{1-e^{-\frac{h}{N}s}}{s}$ appears again, although the reasoning used here is different from that used in the previous subsection. As a matter of fact, $Z$ can be regarded as a generalized holder with a period of $h$ (because the impulse response of $Z$ is non-zero only in $[0,\ h]$).

Furthermore, the coefficients in the series of $H(A)$ can be separated into the sum of a term not involving $s$ and a term including a factor of $s$, i.e.,

$$\begin{aligned} H(A) \;&= \frac{1-e^{-\frac{h}{N}s}}{s} \cdot \Bigg( I + \frac{1}{2!}\frac{h}{N}A + \frac{1}{3!}(\frac{h}{N}A)^2 + \cdots \\ &\quad + (\frac{1 - \frac{s}{1-e^{-\frac{h}{N}s}}\frac{h}{N}e^{-\frac{h}{N}s}}{s} - \frac{1}{2!}\frac{h}{N}) \cdot A \; + (\frac{1}{2!}\frac{2 - \frac{s(2+\frac{h}{N}s)}{1-e^{-\frac{h}{N}s}}\frac{h}{N}e^{-\frac{h}{N}s}}{s^2} - \frac{1}{3!}(\frac{h}{N})^2) \cdot A^2 + \cdots \Bigg) \\ &= \frac{1-e^{-\frac{h}{N}s}}{s} \cdot \left( (e^{\frac{h}{N}A} - I)(\frac{h}{N}A)^{-1} + \hat{H}(A) \right), \end{aligned}$$

where $\hat{H}(A)$ represents the rest of the series in the bracket above. It is easy to show that $\hat{H}(A) = 0$ when $s \to 0$ or $A = 0$. The approximation by the first term $H(A) \approx \frac{1-e^{-\frac{h}{N}s}}{s} \cdot (e^{\frac{h}{N}A} - I)(\frac{h}{N}A)^{-1}$ provides the approximation $Z_{f0}$ given in (13).

### 3.4 Equivalents for the backward rectangular rule

The approximations $Z_f$ and $Z_{f0}$ have an index range of $i = 0, \cdots, N-1$ and hence may be regarded as corresponding to the *forward* rectangular rule. Similar approximations $Z_b$ and $Z_{b0}$, which have an index range of $i = 1, \cdots, N$ and correspond to the *backward* rectangular rule, are

$$Z_b(s) = \frac{1-e^{-s\frac{h}{N}}}{s} \cdot \sum_{i=1}^{N} e^{-i\frac{h}{N}(sI-A)} B, \tag{16}$$

[2]Since $Z(s) = \int_0^h e^{-(sI-A)\theta} d\theta \cdot B$, (14) provides the true value for $Z$. Another way is to use the formula $(1-a)\sum_{i=0}^{N-1} a^i = (1-a^N)$.

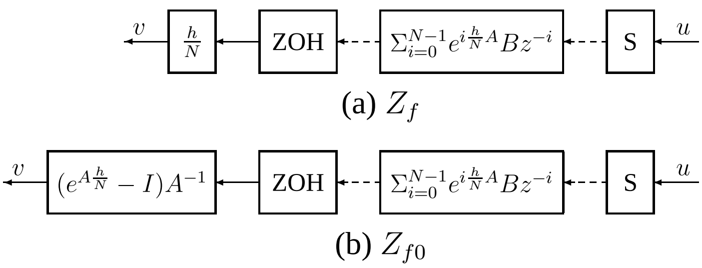



Figure 2: Implementations of $Z$ in the $z$-domain

$$Z_{b0}(s) = \frac{1 - e^{-s\frac{h}{N}}}{s} \cdot \frac{I - e^{-\frac{h}{N}A}}{\frac{h}{N}} A^{-1} \sum_{i=1}^{N} e^{-i\frac{h}{N}(sI-A)} B. \tag{17}$$

As similar to (14), the last formula corresponds to the following quadrature approximation formula:

$$\int_0^h e^{A\zeta} Bu(t-\zeta)d\zeta \approx (I - e^{-\frac{h}{N}A})A^{-1} \cdot \sum_{i=1}^{N} e^{iA\frac{h}{N}} Bu(t - i\frac{h}{N}). \tag{18}$$

When $A = 0$, $(I - e^{-\frac{h}{N}A})A^{-1}$ becomes $\frac{h}{N}$ and hence this new formula can be regarded as an extension of the conventional backward rectangular rule. It provides a better approximation than the conventional backward rectangular rule when the integrand has an exponential term.

$Z_b$ and $Z_{b0}$ hold the properties in Theorems 2 and 3, respectively. It is easy to see that there exists a pure one-step delay $e^{-\frac{h}{N}s}$ in $Z_b$ and $Z_{b0}$ (because $i$ starts from 1 to $N$). It turns out that dropping this term improves the approximation. As a matter of fact, when the pure delay term $e^{-\frac{h}{N}s}$ in $Z_{b0}$ is dropped, $Z_{b0}$ becomes the same as $Z_{f0}$ (see the simulations in Section 6 for accuracy comparison). Hence, the implementation of $Z$ in the next section and the stability issue in Section 5 will be done for $Z_f$ and $Z_{f0}$ only, although some simulations will be given in Section 6 for comparison. The backward rectangular rule is not recommended for implementation of distributed delay.

# 4 Implementation of distributed delay

## 4.1 Implementation in the $z$-domain

The approximations $Z_f$ and $Z_{f0}$, given in (11) and (13), incorporate a hold filter $\frac{1-e^{-\frac{h}{N}s}}{s}$. It is nothing else but a zero-order holder (ZOH), which is an element normally existing in a sampled-data system, and the rest is approximately a polynomial of $z^{-1}$, by using $z \approx e^{s\frac{h}{N}}$. Hence, these transfer functions can be approximately implemented in the $z$-domain with a sampling period of $\frac{h}{N}$, as shown[3] in Figure 2. As pointed out by Kannai and Weiss in [26, Proposition 4.1], the implementations shown in Figure 2 converge to the corresponding transfer functions when $N \to +\infty$. Hence, this approximation step does not change the system stability, provided that $N$ is large enough.

The implementation of $Z_f$ (when ignoring the S and ZOH blocks) looks very similar to $Z_{wf}$ from (12), but there is a significant difference: $Z_{wf}$ in (12) is in the $s$-domain but the implementation in Figure 2 is in the $z$-domain. When $Z$ is implemented in the $z$-domain, the resulting system is a hybrid system and the stability cannot be analyzed by simply replacing the delay term $z^{-1}$ with $e^{-\frac{h}{N}s}$. What has been done here is actually the digital implementation of a continuous time control law. Another way is to re-design a controller for the sampled plant, as reported in [16].

[3] Actually, only the implementation $Z_{f0}$ is needed because $Z_f$ does not guarantee the static gain. It is given here for comparison with $Z_{wf}$ in (12).

## 4.2 Implementation in the $s$-domain

$Z_{f0}$ does not include any hidden unstable poles of the plant, i.e., the eigenvalues of $A$. However, if $Z_{f0}$ is to be implemented in the $s$-domain, extra care has to be taken for the implementation of the hold filter because it includes a hidden unstable pole at $s = 0$. This is more or less the same as the original problem but it is much easier to remove the hidden unstable pole $s = 0$ because the involved delay $\frac{h}{N}$ can be made much shorter than the original delay $h$ (which implies the approximation to be made in a much shorter period, in the sense of the impulse response) and the pole $s = 0$ is known.

The hold filter can be expanded as the following series of $\epsilon$:

$$\frac{1 - e^{-\frac{h}{N}s}}{s} = \frac{1 - e^{-\frac{h}{N}(s+\epsilon)}}{s+\epsilon} + \frac{1 - e^{-\frac{h}{N}(s+\epsilon)} - \frac{h}{N}(s+\epsilon)e^{-\frac{h}{N}(s+\epsilon)}}{(s+\epsilon)^2}\epsilon + \cdots,$$

and hence it can be approximated by the first term as $\frac{1-e^{-\frac{h}{N}s}}{s} \approx \frac{1-e^{-\frac{h}{N}(s+\epsilon)}}{s+\epsilon}$. Here, $\epsilon > 0$ is a small number close to $0$ and hence $\frac{1}{s+\epsilon}$ is stable and implementable. Similarly as before, this approximation does not guarantee the static gain, but the following one does:

$$\frac{1 - e^{-\frac{h}{N}s}}{s} \approx \frac{1 - e^{-\frac{h}{N}(s+\epsilon)}}{s+\epsilon} \frac{\frac{h}{N}\epsilon}{1 - e^{-\epsilon h/N}}. \tag{19}$$

Using this implementation of the hold filter, $Z$ can now be *implemented* in the $s$-domain, corresponding to $Z_{f0}$, as

$$Z_{f\epsilon}(s) = \frac{1 - e^{-\frac{h}{N}(s+\epsilon)}}{1 - e^{-\frac{h}{N}\epsilon}} \frac{e^{\frac{h}{N}A} - I}{s/\epsilon + 1} A^{-1} \cdot \Sigma_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)}B. \tag{20}$$

**Theorem 4.** *The implementation $Z_{f\epsilon}$ holds the following properties: (i) $\lim_{N\to+\infty} Z_{f\epsilon}(s) = Z(s)$; (ii) $Z_{f\epsilon}$ is strictly proper; (iii) $\lim_{s\to 0} Z_{f\epsilon}(s) = \lim_{s\to 0} Z(s)$.*

*Proof.*

$$\lim_{N\to+\infty} \frac{1-e^{-\frac{h}{N}(s+\epsilon)}}{1-e^{-\frac{h}{N}\epsilon}} \frac{1}{s/\epsilon+1} = \lim_{\tau\to 0^+} \frac{1-e^{-\tau(s+\epsilon)}}{1-e^{-\tau\epsilon}} \frac{1}{s/\epsilon+1}$$

$$= \lim_{\tau\to 0^+} \frac{(s+\epsilon)e^{-\tau(s+\epsilon)}}{\epsilon e^{-\tau\epsilon} \cdot (s/\epsilon + 1)} = 1,$$

where the substitution $\tau = h/N$ is used. Hence, $\lim_{N\to+\infty} Z_{f\epsilon}(s) = \lim_{N\to+\infty} Z_{wf}(s) = Z(s)$. The second property is obvious and the last one is easy to prove. This completes the proof. □

*Remark* 2. Here, $\epsilon$ is a small positive number. It can be chosen as close to $0$ as possible whenever it is implementable. However, there is no simple guideline to choose the low-pass filter for the strictly proper implementation proposed in [18]; see the last paragraph of [18, Subsection 4.2]. As to the implementation by adding a low-pass filter proposed in [19], no further suggestions were given for how to choose the low-pass filter.

*Remark* 3. The low-pass filter in the implementations proposed in [18, 19] is added artificially to remedy the instability. The low-pass filter in (20) is inherently there.

# 5 The stability issue related to the implementation

Denote the approximation error of $Z_f$ as

$$E_f = Z_f - Z,$$

and similarly for the other approximation errors. As explained earlier, the approximation errors of the approximations $Z_f$ and $Z_{f0}$ and the implementation $Z_{f\epsilon}$ can be made as small as desirable by choosing a large enough number $N$. Crucially, they are all *strictly proper*. This makes the well-known small-gain theorem (see e.g. [27, 28]) applicable for the stability analysis. Otherwise, a more complicated notion, $w$-stability [29, 30], is needed. Indeed, the following theorem holds.

**Theorem 5.** *The following formulae hold:*

$$\lim_{N\to+\infty} \|E_f(s)\|_\infty = 0,$$

$$\lim_{N\to+\infty} \|E_{f\epsilon}(s)\|_\infty = 0, \qquad (\epsilon \geq 0).$$

*Proof.* According to (11) and (15), $E_f$ is equal to

$$\begin{aligned} E_f(s) &= \left(\frac{1-e^{-s\frac{h}{N}}}{s}I - (I - e^{-(sI-A)\frac{h}{N}})(sI-A)^{-1}\right) \cdot \sum_{i=0}^{N-1} e^{-i\frac{h}{N}(sI-A)}B \\ &= E_1(s)Z_{wf}(s), \end{aligned}$$

where $Z_{wf}$ is as given in (12) and

$$\begin{aligned} E_1(s) &= \frac{N}{h}\left(\frac{1-e^{-s\frac{h}{N}}}{s}I - (I - e^{-(sI-A)\frac{h}{N}})(sI-A)^{-1}\right) \\ &= \frac{N}{h}\int_0^{\frac{h}{N}} e^{-s\tau}(I - e^{A\tau})d\tau. \end{aligned} \tag{21}$$

Since $\|Z_{wf}\|_\infty$ is bounded on the closed right half-plane, it is sufficient to show that $\|E_1\|_\infty$ approaches $0$ when $N \to +\infty$.

$E_1(s)$ is stable and hence it is only needed to consider the convergence on the $j\omega$-axis. It is easy to see from (21) that

$$\|E_1(s)\|_\infty \leq \frac{N}{h}\int_0^{\frac{h}{N}} \left\|I - e^{A\tau}\right\| d\tau.$$

The right side approaches $0$ when $N \to +\infty$, according to the L'Hospital's rule. The second one can be proved similarly. □

*Remark* 4. However, neither $Z_w$ nor $Z_{wf}$ holds this property.

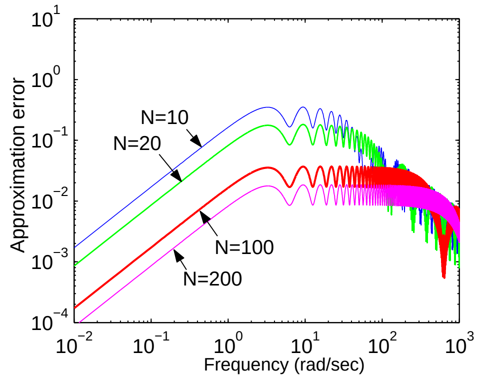
*Remark* 5. It is claimed in [18] that it can be shown the strictly proper implementation proposed there holds this property. However, no proof was given there. There is no proof given in [19] to show that the implementation by adding a low-pass filter hold this property, either.

According to the small-gain theorem, the approximation/implementation error does not cause any instability when $N$ is large enough and there is *no need of any further complicated analysis for the system stability*. Such a need lies in looking for the minimal $N$ to guarantee the system stability. This is a topic left for future research.

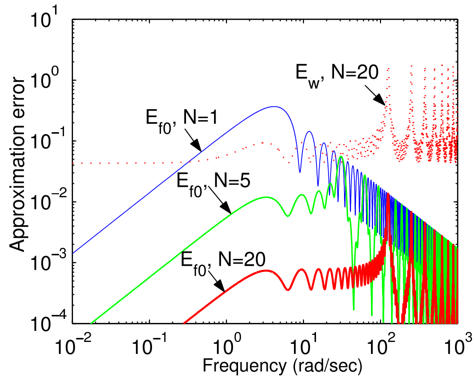
# 6 Numerical examples

Consider the simple plant $\dot{x}(t) = x(t) + u(t-1)$ with the control law

$$u(t) = -(1+\lambda_d)\left(e^1 \cdot x(t) + \int_0^1 e^\zeta u(t-\zeta)d\zeta\right) + r(t). \tag{22}$$

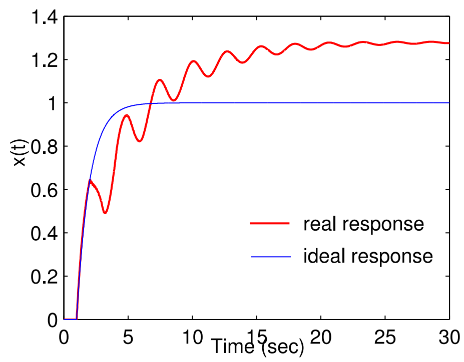



(a) $E_{b0}$

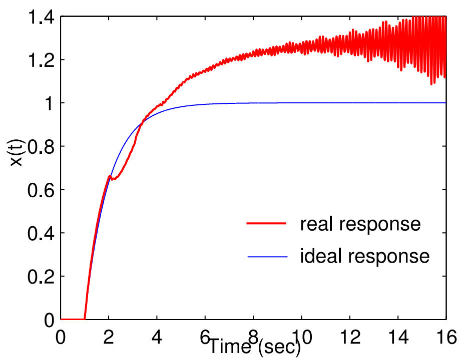



(b) $E_{f0}$ and $E_w$

Figure 3: $E_{b0}$ and $E_{f0}$ for different $N$ and $E_w$ for $N = 20$




(a) $Z$ is implemented as $Z_b$ in the $z$-domain




(b) $Z$ is implemented as $Z_w$ given in (5)

Figure 4: The unit step response ($N = 8$)

This example has been widely studied in the literature; see e.g. [17, 13, 14, 20]. Here, $A = 1$, $B = 1$, $h = 1$ and $F = -(1 + \lambda_d)$. The closed-loop system has only one pole at $s = -\lambda_d$. The closed-loop system is stable when $\lambda_d > 0$. The distributed delay in (22) is

$$v(t) = \int_0^1 e^{\zeta} u(t - \zeta) d\zeta. \tag{23}$$

The ideal implementation $Z$ in the $s$-domain is $Z(s) = \frac{1-e^{1-s}}{s-1}$. The implementation $Z_w$ studied in the literature, as given by (5), is not strictly proper and hence the approximation error, as shown in Figure 3(b) as a dotted line, has a very large magnitude (it does not vanish even when $N \to +\infty$) when the frequency approaches $+\infty$. The proposed approximations are strictly proper. $Z_{b0}$ and $Z_{f0}$ guarantee the static gain of $Z$ (i.e., the error is $0$ at the zero frequency), as can be seen from the frequency responses of $E_{b0}$ and $E_{f0}$ shown in Figures 3(a) and 3(b) for different $N$. The larger the value of $N$, the smaller the approximation error. This verifies that there always exists a number $N$ such that the stability of the closed-loop system is guaranteed. Moreover, for a certain approximation error bound, the number $N$ required by $Z_{f0}$ is much smaller than that required by $Z_{b0}$. $Z_{f0}$ also converges faster than $Z_{b0}$.

In order to keep connection with the results shown in the literature, Figure 4(a) shows the response when $Z$ is implemented in the $z$-domain as $Z_b$ for $N = 8$ and Figure 4(b) shows the response when $Z$ is implemented as $Z_w$. The system is stable when $Z$ is implemented as $Z_b$ but is unstable (as reported) when $Z$ is implemented as $Z_w$. The steady-state behavior of the system has been changed.

Figure 5 shows the implementation error of $Z_{f\epsilon}$ when $N = 1$ for $\epsilon = 1, 0.5, 0.1$ and $0$. The smaller the

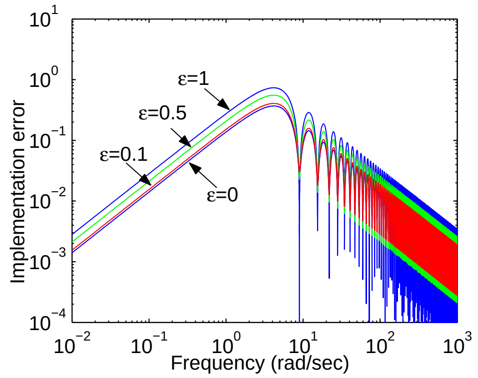



Figure 5: The implementation error of $Z_{f\epsilon}$ for different $\epsilon$ ($N = 1$)

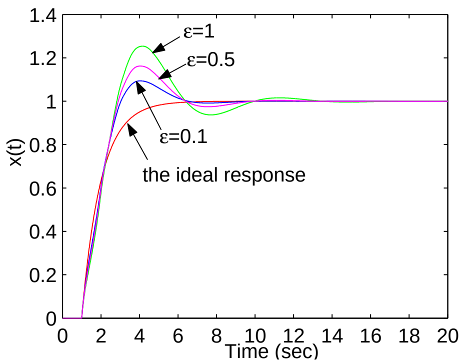



Figure 6: The unit-step response: $Z$ is implemented as $Z_{f\epsilon}$ ($N = 1$)

$\epsilon$, the better the implementation. When $\epsilon = 0.1$, the implementation error is very close to that when $\epsilon = 0$ and there is no need to use an $\epsilon < 0.1$. When $Z$ is implemented in the $s$-domain as this $Z_{f\epsilon}$, i.e.,

$$Z_{f\epsilon}(s) = \frac{e^1 - 1}{1 - e^{-\epsilon}} \frac{1 - e^{-\epsilon} e^{-s}}{s/\epsilon + 1},$$

Figure 6 shows the unit-step response of the system using the control law (22) with $\lambda_d = 1$, i.e.,

$$u = -(1 + \lambda_d) \left( e^1 \cdot x + v \right) + r, \ v = Z_{f\epsilon} \cdot u$$

in the $s$-domain for different $\epsilon$ (note that no change is made to the control law). No instability occurred in the simulations. The steady-state behavior of the system is guaranteed; the transient response is slightly worse than the ideal response, which is due to the approximation of the distributed delay. For different $\epsilon$, the smaller the $\epsilon$ (the better the approximation), the smaller the overshoot. As expected from Figure 5, there is no significant improvement when $\epsilon$ is less than $0.1$.

In summary, the recommended $s$-domain implementation of $Z$ is the $Z_{f\epsilon}$ given in (20) and the $z$-domain implementation is the $Z_{f0}$ shown in Figure 2(b).

# 7 Conclusions

This note proposes two approaches to approximate distributed delay in control laws and then to implement it in the $z$-domain and in the $s$-domain. It is shown that, in the frequency domain, the strict properness of the distributed delay is lost when quadrature approximations are applied. This caused the instability phenomenon reported in the literature. The objective of the proposed approximation is to guarantee the low frequency and the high frequency behaviors of the distributed delay. Moreover, the $H^\infty$-norm of the approximation error converges to $0$ when the number of approximation steps approaches $+\infty$. Hence, the reported instability due to the approximation error disappears, provided that the number $N$ of approximation steps is large enough. The steady-state performance of the system is also guaranteed, without changing the control structure. As by-products, two new formulae for the forward and backward rectangular rules are obtained. These formulae are more accurate than the conventional ones when there is an exponential term in the integrand. Numerical examples are given to verify the proposed results. As shown in simulations, a widely studied system [17, 14, 13], which demonstrated instability, is stable even when $N = 1$.

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