A unified Smith predictor based on the spectral decomposition of the plant *

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Abstract

We point out a numerical problem in the well-known modified Smith predictor and propose a *unified Smith predictor* (USP) which overcomes this problem. The proposed USP combines the classical Smith predictor with the modified one, after a spectral decomposition of the plant. We then derive an equivalent representation of the original delay system, together with the USP. Based on this representation, we give a controller parameterization and we solve the standard H^2 problem.

Index Terms: dead-time compensator, modified Smith predictor, finite-spectrum assignment, finite-impulse-response (FIR) block, Youla parameterization, H^2 problem

1 Introduction

Time delays appear in many physical systems, in particular, those involving material transportation and/or information transmission. Often, systems with delays appear as a simple approximation of more complex infinite-dimensional systems. The classical *Smith predictor* (SP) (Smith,

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1957, 1958) is an effective tool to reduce control problems, such as pole assignment or tracking, for a finite-dimensional LTI stable system with an input or output delay, to corresponding delay-free problems. A finite spectrum assignment scheme was developed in (Olbrot, 1978; Manitius and Olbrot, 1979) to handle input delays in *unstable* plants by state feedback, using the predicted state of the delay system. Watanabe and Ito (1981) overcame some shortcomings in finite-spectrum assignment and other process-model-based control schemes available then, by using a Smith-predictor-like block, which was afterwards called a *modified* (or generalized) *Smith predictor* (MSP), see (Palmor, 1996) and the references therein. Recently, prediction has been recognized as a fundamental concept for the stabilization of delay systems (Mirkin and Raskin, 1999, 2003). Similar predictor blocks have played an important role in H^∞ control of time-delay systems (Meinsma and Zwart, 2000; Zhong, 2003*b,c*; Mirkin, 2003; Zhong, 2003*e*) and in continuous-time deadbeat control (Zhong, 2003*a*; Nobuyama *et al.*, 1996).

We show that the modified Smith predictor may run into numerical problems for delay systems with some fast stable eigenvalues. Indeed, the matrix exponential e^{-Ah} (where h is the delay) appearing in the MSP may be practically non-computable for such systems. Such a numerical problem was mentioned in (Meinsma and Zwart, 2000, p. 279) and a technique, which is not systematic, was suggested in (Zwart *et al.*, 1998) to overcome the problem. This problem was attributed in these papers to large delays. Actually, this numerical problem might occur even for very small delays (with respect to the time constant of the system) if there are some stable eigenvalues λ and they are very fast with respect to the delay h (i.e., the product $|\text{Re}\lambda|\, h$ is large). We propose an alternative predictor, called the *unified Smith predictor* (USP), to overcome this problem. The USP combines the features of the SP and the MSP and it does not require the computation of the matrix exponential for the stable eigenvalues. We achieve this by decomposing the state space of the finite-dimensional part of the plant into unstable and stable invariant subspaces. The controller design techniques based on the MSP have to be re-considered for the USP, to make them applicable in practice. For this reason, we propose an equivalent representation for the *augmented plant*, which consists of the original plant together with the USP. This equivalent representation is then used to give a stabilizing controller parameterization and to solve the standard H^2 problem. Further research is needed to solve the H^∞ control problem for a delay system with a USP.

This paper is organized as follows: The numerical problem with the MSP is explained in Section 2 using a very simple example, and then the USP is introduced. An equivalent representation of the augmented plant is derived in Section 3. We show in Section 4 how this equivalent representation can be used to derive a stabilizing controller parameterization and to solve the H^2 problem.

2 The unified Smith predictor

This paper is written from a frequency domain perspective, which means that LTI systems are represented by their transfer functions. Here, by a transfer function we mean an analytic function defined on a domain which contains a half-plane $\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \mathrm{Re} s > \alpha\}$. A transfer function is called *exponentially stable* if it is bounded on a half-plane $\mathbb{C}_\alpha$ with $\alpha < 0$. Obviously, exponentially stable transfer functions are contained in H^∞. We will use notation (common in the literature)

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{1}$$

meaning that $P(s) = C(sI - A)^{-1}B + D$, where A, B, C, D are matrices.

2.1 The numerical problem with the MSP

Consider a dead-time plant P_h, with $P_h(s) = P(s)e^{-sh}$, where $h > 0$ and the rational part P is realized as in (1). By a *predictor* for P_h we mean an exponentially stable system Z such that $P_h + Z$ is rational. A *predictor-based controller* for the dead-time plant P_h consists of a predictor Z and a *stabilizing compensator* C, as shown in Figure 1 (see also Figure 5 for a more general structure). The underlying idea is the well-known fact that there is a one-to-one correspondence between the set of all the stabilizing controllers for P_h and for $P_h + Z$, see for example Remark 3.6 and Example 4.1 in (Curtain *et al.*, 1996).

In order to simplify the exposition, only the stabilization problem will be studied in this section. If the tracking problem is considered, then an additional constraint on C is required, e.g., if the reference r is a step signal then it is required that $\lim_{s\to 0} C(s)(I+Z(s)C(s))^{-1} = \infty$, see (Watanabe and Ito, 1981; Palmor, 1996) for more details.

If P is stable, then the predictor Z can be chosen to be the *classical Smith predictor* (SP) (Smith, 1957, 1958),

$$Z_{\text{SP}}(s) = P(s) - P(s)e^{-sh}, \tag{2}$$

and the stabilizing compensator C is designed as a stabilizing controller for the delay-free system P. If P is unstable, then the predictor Z can be chosen to be the *modified Smith predictor* (MSP) (Watanabe and Ito, 1981; Palmor, 1996),

$$Z_{\text{MSP}}(s) = P^{\text{aug}}(s) - P(s)e^{-sh}, \tag{3}$$

where $P^{\text{aug}} = P_h + Z_{\text{MSP}}$, the *augmented plant*, is given by

$$P^{\text{aug}} = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & e^{-Ah}B \\ \hline C & 0 \end{array}\right]. \tag{4}$$

Note that Z_{MSP} has its impulse response supported on $[0,\ h]$, hence it is exponentially stable. Now the stabilizing compensator C is designed as a stabilizing controller for P^{aug}. For further background and applications of the MSP we refer to (Meinsma and Zwart, 2000; Mirkin and Raskin, 2003). Implementing Z_{MSP} is a delicate problem, because we have to avoid hidden unstable modes. One approach is to use some form of numerical integration using the past values of u (Palmor, 1996; Manitius and Olbrot, 1979; Zhong, 2003*d*). Alternatively, Z_{MSP} can be approximately implemented by an FIR block with a stable rational part, so that it does not have hidden unstable modes (Watanabe and Ito, 1981). Recently, an interesting implementation of Z_{MSP} using resetting integrators has been proposed in (Mondié *et al.*, 2001).

Now consider a simple example with

$$P = \left[\begin{array}{cc|c} -1000 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 1 & 1 & 0 \end{array}\right],$$

i.e., $P(s) = \frac{1}{s+1000} + \frac{1}{s-1}$. According to (3), the predictor needed is

$$Z_{\text{MSP}}(s) = \frac{e^{1000h} - e^{-sh}}{s+1000} + \frac{e^{-h} - e^{-sh}}{s-1}.$$

Clearly, there is a numerical problem: e^{1000h} is a huge number even for a not so large delay h! Indeed, according to IEEE Standard 754 (IEEE, 1985), which is today the most common

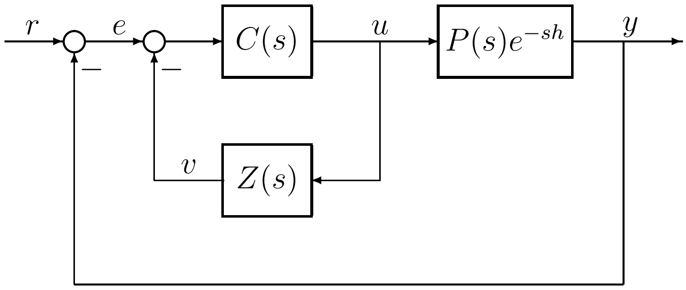

Figure 1: A dead-time plant with a predictor-based controller. The controller consists of the predictor Z (which is exponentially stable) and the stabilizing compensator C. Z is chosen such that the augmented plant $P^{\text{aug}}(s) = Z(s) + P(s)e^{-sh}$ is rational. Then C can be designed as a stabilizing controller for P^{aug}.

representation for real numbers on computers, including Intel-based PC's, Macintoshes and most Unix platforms, this number is regarded to be $+\infty$ (INF) for $h \geq 0.71$sec. Certainly, such a component in a controller is not allowed in a practical application.

We observe that this problem arises due to a fast stable pole (here, $p = -1000$) of P. A stable pole makes the real part of the exponent positive, and if this is large, the numerical problem occurs. If the plant is completely unstable (there are no stable poles), then there will be no such problem. Thus, if the plant has fast stable poles, then the predictor Z_{MSP} from (3) should not be used. An alternative predictor will be introduced below.

2.2 The unified Smith predictor

A natural solution to the numerical problem encountered in the previous subsection is to decompose P into the sum of a stable part P_s and an unstable part P_u and then to construct predictors for them, using the classical Smith predictor (2) for $P_s(s)e^{-sh}$ and the modified Smith predictor (3) for $P_u(s)e^{-sh}$. We propose a new term, *unified Smith predictor* (USP), for such a predictor.

As is well known, the rational part of the plant, P given in (1), can be split (decomposed) into the sum of a stable part P_s and an unstable part P_u with ease, e.g., by applying a suitable linear coordinate transformation in its state space. There exist a lot of such similarity transformations. One of them, denoted here by V, can be obtained by bringing the system matrix A to the Jordan canonical form $J = V^{-1}AV$. (In MATLAB™, this is done by $[\ V \quad J\ ] = \text{jordan}(A)$.) Assume that V is a nonsingular matrix such that

$$P = \left[\begin{array}{c|c} V^{-1}AV & V^{-1}B \\ \hline CV & D \end{array}\right] = \left[\begin{array}{cc|c} A_u & 0 & B_u \\ 0 & A_s & B_s \\ \hline C_u & C_s & D \end{array}\right], \tag{5}$$

where A_s is stable and A_u is completely unstable. As a matter of fact, the decomposition can be done by splitting the complex plane along any vertical line Re $s = \alpha$ with $\alpha \leq 0$. Then the eigenvalues of A_u are all the eigenvalues λ of A with Re $\lambda \geq \alpha$, while A_s has the remaining eigenvalues of A. In the sequel, in order to simplify the exposition, we use the imaginary axis ($\alpha = 0$) to split the complex plane. Now P can be split as $P = P_s + P_u$, where

$$P_s = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & 0 \end{array}\right] \qquad \text{and} \qquad P_u = \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D \end{array}\right].$$

The predictor for the stable part P_s can be taken as a classical SP,

$$Z_s(s) = P_s(s) - P_s(s)e^{-sh}, \tag{6}$$

and the predictor for the unstable part P_u can be taken as the following MSP:

$$Z_u(s) = P_u^{\text{aug}}(s) - P_u(s)e^{-sh}, \qquad P_u^{\text{aug}} \doteq \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u e^{-A_u h} & 0 \end{array}\right].$$

Then the USP for the plant $P_h(s) = P(s)e^{-sh}$ is defined by $Z = Z_s + Z_u$, as shown in Figure 2. It is now clear that

$$Z(s) = P^{\text{aug}}(s) - P(s)e^{-sh}, \tag{7}$$

where $P^{\text{aug}} = P_s + P_u^{\text{aug}}$ and a realization of P^{aug} is

$$P^{\text{aug}} = \left[\begin{array}{c|c} A & B \\ \hline CE_h & 0 \end{array}\right] = \left[\begin{array}{c|c} A & E_h B \\ \hline C & 0 \end{array}\right], \qquad E_h = V \begin{bmatrix} e^{-A_u h} & 0 \\ 0 & I_s \end{bmatrix} V^{-1}. \tag{8}$$

Here, the identity I_s has the same dimension as A_s and we have used that $E_h A = A E_h$. The impulse response of the USP is not finite, unlike for the MSP. The above P^{aug} is the *augmented plant*, obtained by connecting the original plant and the USP in parallel. The stabilizing compensator C in Figure 1 should be designed as a stabilizing controller for P^{aug}.

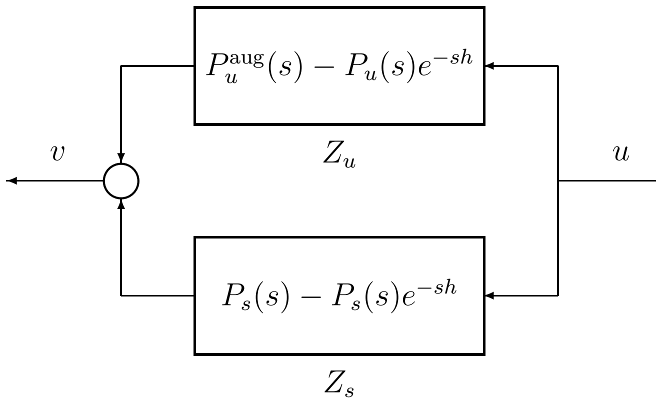

Figure 2: The unified Smith predictor $Z = Z_s + Z_u$. Here, Z_s is a classical SP for P_s, the stable part of P, and Z_u is an MSP for P_u, the unstable part of P.

The use of the different predictors (SP, MSP and USP) is summarized in Table 1: when the plant is stable, the USP reduces to the SP; when the plant is completely unstable, the USP reduces to the MSP; when the plant is unstable but has some stable poles (mixed), then the USP should be used, especially if some stable poles are fast (as explained earlier).

Type of plant	stable	completely unstable	mixed
Type of USP	SP	MSP	USP

Table 1: The type of USP needed for different types of plants

Returning to the example at the beginning of this section, the USP for it is given by

$$Z(s) = \frac{1 - e^{-sh}}{s + 1000} + \frac{e^{-h} - e^{-sh}}{s - 1}.$$

Remark 1. A possible realization of Z consists of the finite-dimensional system

$$\begin{aligned} \dot{z}(t) &= Az(t) + E_h Bu(t) - Bu(t-h) \\ v(t) &= Cz(t) - Du(t-h) \end{aligned}$$

together with a realization of a delay line (which is needed to generate the signal $u(t-h)$). However, such a realization of Z would have hidden unstable modes. An alternative is to realize Z_s and Z_u separately. The first is not a problem (it can be implemented dynamically), while for Z_u see the comments after (4) and Remark 4.

Remark 2. Z_s defined in (6) could be replaced by

$$Z_s(s) = -P_s(s)e^{-sh},$$

as in internal model control (Morari and Zafiriou, 1989). In this case, P^{aug} in (8) would change, since in the definition of E_h we would have to replace the identity I_s by zero. Now P^{aug} would be simpler (it would have a lower order). This may be advantageous in the context of Figure 1. However, in the more general framework of Figure 5 (in the next section) this may cause a rank problem (as we shall see).

Remark 3. The USP (in particular, the SP and the MSP) can be generalized to multiple delays, i.e., to the situation where the components u_1, u_2, $\ldots$ u_m of the vector u are delayed by h_1, h_2, $\ldots$ $h_m \geq 0$. Denote $H = \mathrm{diag}(h_1,\, h_2,\, \ldots\, h_m)$ and

$$E \otimes B = \begin{bmatrix} E_{h_1} b_1 & E_{h_2} b_2 & \ldots & E_{h_m} b_m \end{bmatrix},$$

where b_k is the k-th column of B, i.e., $B = \begin{bmatrix} b_1 & b_2 & \ldots & b_m \end{bmatrix}$. Then

$$P^{\mathrm{aug}} = \left[\begin{array}{c|c} A & E \otimes B \\ \hline C & 0 \end{array}\right], \qquad Z(s) = P^{\mathrm{aug}}(s) - P(s)e^{-Hs} \tag{9}$$

and Z is exponentially stable.

Remark 4. We outline an implementation of Z from Remark 3 (the USP for multiple delays) which avoids the problem of unstable modes. We use the notation from (5). We realize the components Z_u and Z_s separately, but use the same delay lines for both, see Figure 3. The component Z_u is a hybrid system, containing two copies of an LTI block with the possibility of resetting, and the switches S_a, S_b and S_v. In this diagram, we have denoted $B_u = \begin{bmatrix} \beta_1 & \beta_2 & \ldots & \beta_m \end{bmatrix}$ and $E \otimes B_u = \begin{bmatrix} e^{-A_u h_1}\beta_1 & e^{-A_u h_2}\beta_2 & \ldots & e^{-A_u h_m}\beta_m \end{bmatrix}$. The output of the USP is $v = v^s + v^u$, where $v^s = Z_s\, u$ and $v^u = Z_u\, u$. v^u is produced by one of the two LTI blocks which can be reset by the signals R_a and R_b, respectively. When we reset one of them, say the top one, at a time τ, we also switch its second input u^a from u^d to 0 using the switch S_a. Afterwards, we reconnect u^a_k to u^d_k at the time $\tau + h_k$ ($k = 1, 2, \ldots m$). During this time, $v^u = v^b$. After all the components of u^d have been reconnected to the upper LTI block, so that $u^a = u^d$, its output v^a will be the desired value of v^u, and so the switch S_v can be set so that $v^u = v^a$. The switches could, theoretically, remain in this position forever, but since A_u is unstable, tiny computational or rounding errors or the effect of noise will grow and will corrupt the output v^a and hence v^u. To prevent this, we reset the other LTI block (while switching the components of u^b, similarly as above, using the switch S_b) and, when the output v^b becomes correct, we switch S_v so that $v^u = v^b$. This cycle repeats itself indefinitely and it can be shown that the USP is stable and produces the correct output. This implementation is related to the resetting Smith predictor of (Mondié *et al.*, 2001).

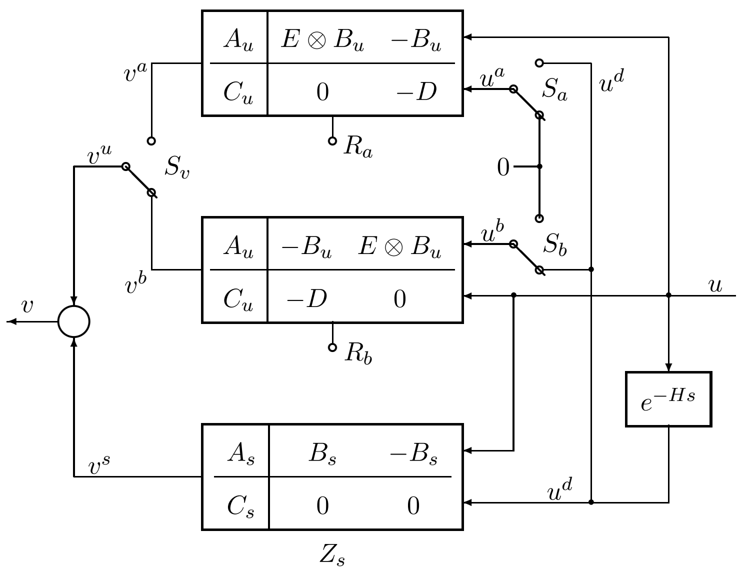

Figure 3: An implementation of the USP (9) for multiple delays, using two resetting LTI systems to implement the component Z_u. The logical controller which controls the switches and generates the resetting signals R_a and R_b (in an open loop manner) is not shown.

3 Control systems with a USP – equivalent diagrams

In this section we consider a more general type of plant, with two inputs and two outputs (each of these signals may be vector-valued). The input signal w contains references and disturbances, u is the control input, z is the tracking error and y is the measurement available to the controller. Such a plant is commonly considered in robust control, see for example (Green and Limebeer, 1995; Zhou *et al.*, 1996). We assume that the plant P_h consists of a rational part P and a delay by $h > 0$ acting on u, as shown in Figure 4. We denote

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{10}$$

K is called a *stabilizing controller* for P_h if the transfer functions from the three external inputs shown in Figure 4 to any other signal in the diagram are in H^∞. K is called an *exponentially stabilizing controller* for P_h if the same transfer functions are exponentially stable. In the sequel, the two external signals appearing in the lower part of Figure 4 will be taken to be zero.

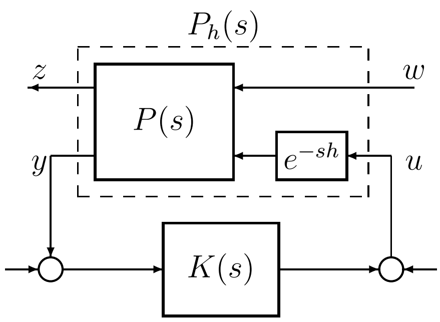

Figure 4: The control system comprising a dead-time plant P_h (with a rational part P) and a stabilizing controller K. While P may be unstable, we would like the control system to be (exponentially) stable and the transfer function from w to z to be small.

As in the previous section, let V be a nonsingular matrix such that

$$V^{-1}AV = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix},$$

where A_u is completely unstable and A_s is stable. As in (7), we introduce the following USP designed for the component P_{22} of the plant:

$$Z(s) \doteq P_{22}^{\mathrm{aug}}(s) - P_{22}(s)e^{-sh}, \tag{11}$$

where, as in (8),

$$P_{22}^{\mathrm{aug}} \doteq \left[\begin{array}{c|c} A & B_2 \\ \hline C_2E_h & 0 \end{array}\right], \qquad E_h = V \begin{bmatrix} e^{-A_uh} & 0 \\ 0 & I_s \end{bmatrix} V^{-1}.$$

By connecting this USP in parallel with the u to y component of P_h, as shown in Figure 5, thus creating a new measurement output y_p, we obtain a new *augmented plant*

$$P^{\text{aug}}(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s)e^{-hs} \\ P_{21}(s) & P_{22}^{\text{aug}}(s) \end{bmatrix}. \tag{12}$$

It is well known (and easy to see from Figure 5) that C is a stabilizing controller for P^{aug} if and only if $K = C(I - ZC)^{-1}$ is a stabilizing controller for the original dead-time plant. The same statement remains true with "exponentially stabilizing" in place of "stabilizing" (and, of course, the corresponding set of controllers is smaller).

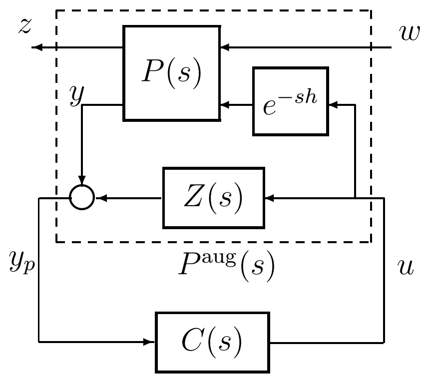

Figure 5: The control system from Figure 4, in which the controller K has been decomposed into a USP denoted Z and a stabilizing compensator C, so that $K = C(I - ZC)^{-1}$. The augmented plant P^{aug} consists of the plant P_h together with the USP.

We derive now an equivalent representation of P^{aug}, which is useful for the problems treated in the next section. Proposition 1 and Remark 5 below are related to Lemma 1 in (Mirkin and Raskin, 2003) (the corresponding notation is $\Delta_1 = Z_1$ and $\Delta_2 = Z$).

Proposition 1. *We have*

$$P^{\text{aug}}(s) = \begin{bmatrix} Z_1(s) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} e^{-sh}I & 0 \\ 0 & I \end{bmatrix} \tilde{P}(s), \tag{13}$$

where

$$Z_1(s) \doteq P_{11}(s) - \left[\begin{array}{c|c} A & e^{Ah}B_1 \\ \hline C_1 & 0 \end{array}\right](s)e^{-sh},$$

$$\tilde{P} = \tilde{P}_0 + \begin{bmatrix} \tilde{P}_s & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{P}_0 = \left[\begin{array}{c|cc} A & E_h^{-1}B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2E_h & D_{21} & 0 \end{array}\right], \tag{14}$$

$$\tilde{P}_s = \left[\begin{array}{c|c} A_s & \begin{bmatrix} 0 & e^{A_s h} - I_s \end{bmatrix} V^{-1}B_1 \\ \hline C_1 V \begin{bmatrix} 0 \\ I_s \end{bmatrix} & 0 \end{array}\right]. \tag{15}$$

The block diagram corresponding to the decomposition (13)-(14) of P^{aug} is in Figure 6.

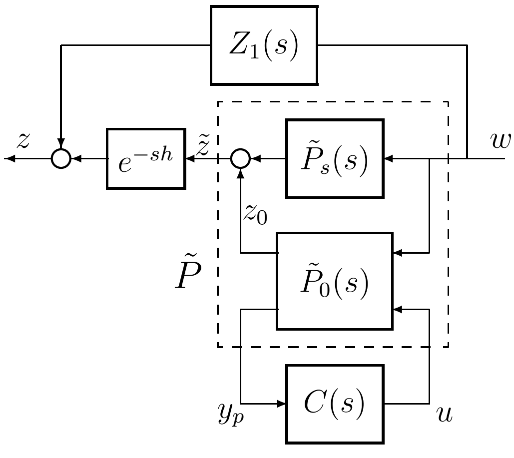

Figure 6: An equivalent representation of the control system in Figure 5, using the decomposition of P^{aug} given in Proposition 1. Here, Z_1 and $\tilde{P}_s$ are exponentially stable.

Proof. Using the partition (12) of P^{aug}, the formula (13) (which we have to prove) is equivalent to the following four formulas:

$$P_{11}(s) = Z_1(s) + \left(\left[\begin{array}{c|c} A & E_h^{-1}B_1 \\ \hline C_1 & 0 \end{array}\right](s) + \tilde{P}_s(s) \right) e^{-sh}, \tag{16}$$

$$P_{12}(s)e^{-sh} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right](s)e^{-sh},$$

$$P_{21} = \left[\begin{array}{c|c} A & E_h^{-1}B_1 \\ \hline C_2E_h & D_{21} \end{array}\right], \qquad P_{22}^{\text{aug}} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2E_h & 0 \end{array}\right].$$

The last three formulas are clearly true, so only the first formula (16) requires a little work. If we rewrite

$$\left[\begin{array}{c|c} A & E_h^{-1}B_1 \\ \hline C_1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_u & 0 & \multirow{2}{*}{$\begin{bmatrix} e^{A_u h} & 0 \\ 0 & I_s \end{bmatrix} V^{-1}B_1$} \\ 0 & A_s & \\ \hline \multicolumn{2}{c|}{C_1 V} & 0 \end{array}\right]$$

and if we rewrite

$$\tilde{P}_s = \left[\begin{array}{cc|c} A_u & 0 & \multirow{2}{*}{$\begin{bmatrix} 0 & 0 \\ 0 & e^{A_s h} - I_s \end{bmatrix} V^{-1}B_1$} \\ 0 & A_s & \\ \hline \multicolumn{2}{c|}{C_1 V} & 0 \end{array}\right],$$

then (16) follows easily. □

Note that if we would choose the USP described in Remark 2, then $E_h$would become singular and hence the decomposition (14) of $\tilde{P}$ would have to be replaced by a more complicated one. This is the rank problem mentioned in Remark 2.

Remark 5. It follows from Proposition 1 that the closed-loop transfer function T_{zw} from w to z can be written as the sum of three terms:

$$T_{zw}(s) = Z_1(s) + \tilde{P}_s(s)e^{-sh} + \mathcal{F}_l(\tilde{P}_0(s), C(s))e^{-sh}, \tag{17}$$

where $\mathcal{F}_l(\tilde{P}_0, C)$ is the transfer function from w to z_0 in Figure 6. Z_1 is an FIR system with impulse response supported on $[0, h]$, while the second and the third term have impulse responses supported on $[h, \infty)$. Hence, if these terms are in $H^2(\mathbb{C}_0)$, then the first term is orthogonal to the second and third term.

Remark 6. A realization of $\tilde{P}$ is given by

$$\tilde{P} = \left[\begin{array}{cc|cc} A & 0 & E_h^{-1}B_1 & B_2 \\ 0 & A_s & \begin{bmatrix} 0 & e^{A_s h} - I_s \end{bmatrix} V^{-1}B_1 & 0 \\ \hline C_1 & C_1 V \begin{bmatrix} 0 \\ I_s \end{bmatrix} & 0 & D_{12} \\ C_2 E_h & 0 & D_{21} & 0 \end{array}\right]. \tag{18}$$

If P is completely unstable, then the dimensions of A_s and I_s are zero and the block $\tilde{P}_s$ in

Figure 6 disappears. In this case, Z reduces to an MSP and $\tilde{P}$ becomes

$$\tilde{P} = \left[\begin{array}{c|cc} A & e^{Ah}B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 e^{-Ah} & D_{21} & 0 \end{array}\right] = \tilde{P}_0.$$

If P is stable, then $\tilde{P}$ is reduced to

$$\tilde{P} = \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & A & (e^{Ah} - I)B_1 & 0 \\ \hline C_1 & C_1 & 0 & D_{12} \\ C_2 & 0 & D_{21} & 0 \end{array}\right].$$

4 Applications

4.1 The parameterization of all stabilizing controllers

We consider the control system in Figure 4 and we derive necessary and sufficient conditions on P_h for the existence of an exponentially stabilizing controller K. It is known that in general, exponential stabilization for an infinite-dimensional plant is more difficult to achieve than stabilization (in the H^∞ sense) (Weiss *et al.*, 2001). It turns out that in our particular setting, the two problems are equivalent, and can be reduced to the stabilization of $\tilde{P}_0$ from (14).

Theorem 2. *The dead-time plant P_h shown in Figure 4 with a minimal realization of its rational part as in (10) admits an (exponentially) stabilizing controller K (as defined in Section 3) if and only if (A, B_2) is stabilizable and (C_2, A) is detectable. With the USP Z given in* (11)*, every such controller can be expressed as*

$$K = C(I - ZC)^{-1}, \tag{19}$$

where C is an (exponentially) stabilizing controller for $\tilde{P}_0$ defined in (14). Let F and L be such that $A + LC_2$ and $A + B_2F$ are stable, then every stabilizing C for $\tilde{P}_0$ can be expressed as

$$C = \mathcal{F}_l(M, Q), \qquad M = \left[\begin{array}{c|cc} A + B_2F + E_h^{-1}LC_2E_h & -E_h^{-1}L & B_2 \\ \hline F & 0 & I \\ -C_2E_h & I & 0 \end{array}\right] \tag{20}$$

where $Q \in H^\infty$. Such a C is exponentially stabilizing if and only if Q is exponentially stable. The closed-loop transfer function T_{zw} achieved by K from (19) is given by (17), where

$$\mathcal{F}_l(\tilde{P}_0, C) = \mathcal{F}_l(N, Q),$$

$$N = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & E_h^{-1}B_1 & B_2 \\ 0 & A + E_h^{-1}LC_2E_h & E_h^{-1}(B_1 + LD_{21}) & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & 0 & D_{12} \\ 0 & C_2E_h & D_{21} & 0 \end{array}\right].$$

Proof. As explained in the previous section, the control system in Figure 4 has an equivalent representation shown in Figure 6. The blocks Z_1 and $\tilde{P}_s$ in Figure 6 are exponentially stable, so that they have no influence on the (exponential) stability of the whole system (since they are not a part of any feedback loop). Thus, the (exponential) stability of the original closed-loop system is equivalent to that of the system formed from $\tilde{P}_0$ and C only.

First we discuss stabilization in the H^∞ sense. Since $\tilde{P}_0$ is rational, the parameterization of the stabilizing controllers using M follows from (Zhou *et al.*, 1996, p. 312) where we replace C_2 by C_2E_h and L by $E_h^{-1}L$. This is possible because $A + LC_2$ is similar to $A + E_h^{-1}LC_2E_h$ (hence, $A + LC_2$ is stable if and only if $A + E_h^{-1}LC_2E_h$ is stable). For a rigorous derivation of the parameterization for irrational plants, see (Curtain *et al.*, 2001). The formula involving N follows from (Zhou *et al.*, 1996, p. 323).

Now we consider the exponential stabilization. Let $\alpha < 0$ be such that all the eigenvalues of $A + B_2F$ and $A + LC_2$ are in the half-plane where Re $s < \alpha$. Now all the arguments used to derive the parameterization (20) of stabilizing controllers can be redone with H^∞ replaced by $H^\infty(\mathbb{C}_\alpha)$, the space of bounded analytic functions on $\mathbb{C}_\alpha$. This will result in the same formulae (20), but now Q must be in $H^\infty(\mathbb{C}_\alpha)$. □

4.2 The H^2 problem

Consider again the feedback system from Figure 4. The H^2 problem is to find a stabilizing controller K which minimizes the H^2-norm of the transfer function T_{zw}. The solution of this problem, using the modified Smith predictor, is well known, see (Mirkin and Raskin, 1999,

2003) and the references therein. In this section, we will re-consider this problem using the USP. We shall work with a minimal realization of P, of the form (10). Assume the following:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$;

(A3) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R}$;

(A4) $D_{12}^* D_{12} = I$ and $D_{21} D_{21}^* = I$.

Assumption (A4) is made just to simplify the exposition. In fact, only the non-singularity of $D_{12}^* D_{12}$ and $D_{21} D_{21}^*$ is required (Green and Limebeer, 1995; Zhou and Doyle, 1998).

As mentioned in Remark 5, the impulse response of Z_1 is supported on $[0, h]$ and the last two terms of T_{zw} in (17) (which together are $\mathcal{F}_l(\tilde{P}, C)$ delayed by h) are supported on $[h, \infty)$. Assume for a moment that $D_{11} = 0$. Then it follows that Z_1 is orthogonal (in H^2) to the last two terms of T_{zw}, so that

$$\begin{aligned} \|T_{zw}\|_2^2 &= \|Z_1\|_2^2 + \left\| \mathcal{F}_l(\tilde{P}, C) e^{-sh} \right\|_2^2 \\ &= \|Z_1\|_2^2 + \left\| \mathcal{F}_l(\tilde{P}, C) \right\|_2^2 . \end{aligned}$$

The H^2 control problem of minimizing $\|T_{zw}\|_2$ over all stabilizing K is then converted to

$$\gamma = \min_C \left\| \mathcal{F}_l(\widetilde{P}, C) \right\|_2 \qquad (C \text{ stabilizing}).$$

This is a finite-dimensional H^2 problem which can be solved using known results, see e.g. (Zhou *et al.*, 1996, Theorem 14.7 on p. 385). This last problem is meaningful even if $D_{11} \neq 0$: we are then minimizing the L^2 norm of the impulse response of T_{zw} restricted to $[h, \infty)$. Note that the restriction to $[0, h]$ is independent of C. Rewrite $\tilde{P}$ from (18) as

$$\tilde{P} = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & 0 & D_{12} \\ \tilde{C}_2 & D_{21} & 0 \end{array} \right],$$

then the solution of the H^2 problem involves the following two Hamiltonian matrices:

$$H_2 = \begin{bmatrix} \tilde{A} & 0 \\ -\tilde{C}_1^* \tilde{C}_1 & -\tilde{A}^* \end{bmatrix} - \begin{bmatrix} \tilde{B}_2 \\ -\tilde{C}_1^* D_{12} \end{bmatrix} \begin{bmatrix} D_{12}^* \tilde{C}_1 & \tilde{B}_2^* \end{bmatrix},$$

$$J_2 = \begin{bmatrix} \tilde{A}^* & 0 \\ -\tilde{B}_1\tilde{B}_1^* & -\tilde{A} \end{bmatrix} - \begin{bmatrix} \tilde{C}_2^* \\ -\tilde{B}_1 D_{21}^* \end{bmatrix} \begin{bmatrix} D_{21}\tilde{B}_1^* & \tilde{C}_2 \end{bmatrix}.$$

When the conditions (A1-A4) hold, these two Hamiltonian matrices belong to $dom(Ric)$, as defined in (Zhou *et al.*, 1996), and moreover, $X_2 \doteq Ric(H_2) \geq 0$ and $Y_2 \doteq Ric(J_2) \geq 0$.

Theorem 3. *There exists a unique optimal H^2 controller*

$$K = C(I - ZC)^{-1}, \qquad C = \left[\begin{array}{c|c} \tilde{A} + \tilde{B}_2 F_2 + L_2\tilde{C}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right],$$

where

$$F_2 = -(\tilde{B}_2^* X_2 + D_{12}^*\tilde{C}_1), \quad L_2 = -(Y_2\tilde{C}_2^* + \tilde{B}_1 D_{21}^*).$$

Proof. The only thing to be verified is whether the conditions (A1)-(A4) hold for $\tilde{P}$. Using the zeros in the matrix in (18), we can see that this is true, since A_s is stable. The solution can be obtained by applying Theorem 14.7 from (Zhou *et al.*, 1996, p. 385). □

5 Discussions and conclusions

We have pointed out a numerical problem with the modified Smith predictor when the plant has fast stable poles and we have introduced the unified Smith predictor as a remedy. We have derived an equivalent representation of the augmented plant consisting of a dead-time plant and a unified Smith predictor. Using this representation, we have derived a parameterization of the (exponentially) stabilizing controllers for the dead-time plant (with the USP connected to it) and we have solved the standard H^2 problem (again, in the presence of the USP).

The same numerical problem may arise also in the solution of the H^∞ control problem for dead-time plants (Meinsma and Zwart, 2000; Zhong, 2003*b*; Mirkin, 2003; Zhong, 2003*e*), where the controller design involves the computation of e^{-Ah}. A reasonable assumption is that the computation of e^{Ah} does not cause such a problem (due to the unstable modes in A), because otherwise the system "blows up" before the control can have any effect.

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