Equiripple Minimum Phase FIR Filter Design From Linear Phase Systems Using Root Moments

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***Abstract*—In this brief we propose to design a minimum phase finite-impulse response (FIR) digital filter transfer function from a given equiripple linear phase FIR transfer function which has identical amplitude. The brief deals with two important issues. The first issue is that we are concentrating on very high degree polynomials for which factorization procedures for root extraction are unreliable. A novel approach is taken, that involves the use of a set of parameters called root moments of a polynomial, derivable directly from the polynomial coefficients that immensely facilitate the factorization of polynomials of very large order. The second issue is that the polynomials we deal with have roots on the unit circle. In the paper we propose a method to overcome this problem. The results of the proposed design scheme are very encouraging as far as robustness and computational complexity are concerned.**

***Index Terms*—Group delay, linear phase filters, minimum phase filters, Reméz exchange algorithm, root moments.**

I. Introduction

Finite-impulse response (FIR) digital filters have been extensively studied in the past years for several reasons. Among these are the properties of stability and high-speed implementation using the fast Fourier transform (FFT) or other number-theoretic transforms. Discussions of some of the most popular design methods for FIR filters, and references to some of the literature, are contained in [1]–[4]. We concentrate on real FIR transfer functions and in this context we can classify them into three main categories: 1) linear phase (LP), in which all passband zeros are in conjugate reciprocal pairs around the unit circle; 2) minimum phase (MP), where all passband zeros are inside the unit circle; and 3) a general nonlinear, nonMP category, where neither of the first two conditions hold.

However, most of the work done is concentrated on LP filters [2]. This may be due, in part, to the availability of efficient FIR linear phase filter design algorithms which rely on the formulation of optimization methods for the design of a desired magnitude response [2]. For example, a class of optimization methods for designing FIR filters [2] are based on the assumption that the filter coefficients are symmetrical (or antisymmetrical) and this implies that the filter is LP as already mentioned. The research efforts that have been devoted to the LP case have resulted in powerful and efficient computer algorithms (e.g., the widely used computer program by McClellan *et al.* [3]). In general, we can say that design algorithms for the MP case tend to be significantly more complicated than those for LP.

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The main advantage of LP filters is constant time delay over the entire band. This is a useful property in certain applications such as data transmission, which requires a nondispersive channel to avoid problems such as intersymbol interference (ISI). The most popular existing method for the design of LP FIR filters is the Reméz exchange algorithm [1], [2].

However, MP FIR filters are also known to have certain advantages. The MP filter exhibits a very significant saving in group delay for high orders, compared with the LP equivalent. It also exhibits reduced order for given gain specifications, and lower coefficient sensitivity to quantization error, when compared with the more common LP filters.

In this study we are focusing on applications where the LP characteristic may not be necessary. In fact, the associated excess group delay for such filters may even be undesirable for many applications. A representative example is in telecommunications where long time delays increase the problems of echo and singing on voice lines. The solution to achieving lower time delay is the implementation of a MP design.

Another application is speech processing. The fact that in speech channels phase dispersion is of less importance since the ear is rather insensitive to phase distortion, is a very important issue that stimulates efforts for MP FIR filter designs.

Therefore, a MP design in order to reduce group delay through the filter is of fundamental importance.

There is a quite significant amount of work concerning the design of MP FIR filters. Below we summarize the most representative contributions in the area.

In 1976, Holt *et al.* [9] demonstrated how to use the Reméz exchange algorithm [1] in order to approximate both amplitude and phase response of a nonlinear phase filter. The amplitude response performance of the resulting filter is somewhat worse [10] than of optimal (i.e., minimal order) LP filters.

Herrmann and Schuessler [11] have presented a method of transforming equiripple LP FIR filters into MP FIR filters with half the degree and the same type of attenuation characteristics in the modulus squared. Their technique has been extended by Burris [12] to more general filter specifications. When this technique is used to transform multiband or nonequiripple filters, the optimality of the filter is not necessarily maintained. Moreover, using this technique, it is difficult to impose *a priori* design specifications such as the ratio of the passband and stopband ripple of the resulting MP filter. Nevertheless for a given particular LP filter, *a posteriori*, the ratio and ripple of the MP filter, which would result upon transformation, can be calculated.

An alternative approach for designing a MP filter was also suggested by Burris [13], using the relation between gain and phase response of MP filters. It appears that MP filters, designed using the last approach, need fewer coefficients than optimal LP filters with the same gain response specifications. In his specific example, when changing from the LP filter to the MP equivalent the number of coefficients is reduced from 25 to 15, However, Burris gives just one example, for a oneband shaping filter, in which the response is not equiripple and thus may also be suboptimum.

In [14] a method is presented by Gilchrist, which can be used for the direct design of MP FIR digital fillers with arbitrary prescribed modulus-squared characteristics. This method uses the Fletcher-Powell optimization technique to minimize an L_{2p} norm to obtain a nonnegative polynomial fit to the desired characteristics in a transformed domain. The coefficients of the filter are then obtained from the zeros of this polynomial obtained through normal factorization processes. Examples of low-pass and band-pass filter designs are given using an L_{20} norm. A comparison of the passband and stopband ripples and their ratio is made with LP filter designs. However, Gilchrist shows that an equiripple MP filter and a LP filter designed using the Reméz exchange algorithm, with the same order and frequency band edges, have the same deviations. Hence the Gilchrist solution, although equiripple, cannot be optimum.

In [15] Foxal *et al.* compare the calculated and measured response of a MP CCD low-pass transversal filter with that for a LP design with the same magnitude characteristics. It is shown that the MP design can offer up to an order of magnitude improvement in group delay, and is also less sensitive to transfer inefficiency and tap-weight error than the LP design. The two approaches, linear and MP design are compared on the basis of group delay and sensitivity to change transfer inefficiency and coefficient error.

In [16], Goldberg *et al.* discuss some aspects of a design method similar to the one proposed by Herrman *et al.* [11]. First, the prototype LP filter is designed using more efficient design methods than the one used in [11]. Then, a discussion is given on the problem of defining the specifications on the prototype LP filter. Next, the performance of the resulting NLP filter is theoretically and practically compared to that of the optimal (minimal order) LP filter. Finally, some practical limitations in using the proposed design method are brought up, and an alternative design method is proposed in order to avoid these limitations. However only MP filters are obtainable and a factorization procedure is required.

In [17] a procedure for circumventing the difficult problem of locating the roots of a (generally) high-order polynomial accurately is proposed by Mian and Nainer. This is based on the work presented in [18] by Schmidt and Rabiner. To this purpose, it is shown that the use of a basic property of the complex cepstrum reduces the factorization problem (or, equivalently in the time domain, the deconvolution problem) essentially to the computation of two FFT's. Examples are given which illustrate the proposed procedure. The authors are inspired from the work done in [19], where it is necessary to solve for the roots of a mirror-image polynomial and to factorize them suitably, in order to obtain the MP counterpart. The algorithm demands 1) a LP prototype; 2) the determination of the zeros' configuration of that prototype; and 3) the remapping of the zeros, using a numerical search technique to achieve an optimum MP filter.

In [20] Chit and Mason proposed a new adaptive design algorithm, extending earlier work of the same authors on optimum LP filters with *a priori* constrained (anti)symmetric impulse responses [21].

In [22] (a study by Rasmussen and Etter following the work done in [23], [24]), the design algorithm requires a gain specification in a manner identical to the LP case. Conditions for optimality are investigated in terms of the zeros' configuration, equiripple characteristics, minimax and LMS criteria. The result is a very flexible MP FIR filter design technique, with the potential to accommodate a number of different filter structures and types. In [22] the authors treat the MP FIR estimate of an unknown system as a constrained optimization problem in the sense that the zeros of the model must be constrained to be within the unit circle. The authors compare several structures for adaptively obtaining the optimum FIR MP filter model for an unknown FIR system.

In our proposed method a new approach for polynomial factorization without root finding is attempted. The approach taken involves the use of the Cauchy Residue Theorem applied to the contour integral of the logarithmic derivative of the transfer function around appropriate curves. This leads into a set of parameters, the root moments, derivable directly from the polynomial coefficients that facilitate the factorization problem. In addition, the results of the proposed design scheme are very encouraging as far as robustness and computational complexity are concerned.

II. BACKGROUND

A. LP FIR Filter Transfer Functions

We consider a LP real FIR digital filter transfer function having the following form:

$$H(z) = z^n + h_1 z^{n-1} + h_2 z^{n-2} + \cdots + h_n = \prod_{i=1}^{n} (z - r_i). \quad (1)$$

It can be proven [1], that an FIR filter has LP if its unit sample response satisfies the symmetry or the anti-symmetry condition shown below

$$h_i = \pm h_{n-1}, \qquad i = 0, 1 \ldots, n.$$

In that case, the impulse response is called symmetric if the "+" holds or anti-symmetric otherwise. The above condition implies [1] that if a given LP FIR filter transfer function has a zero at the location r_i, it will also have zeros at locations $1/r_i$, r_i^* and $1/r_i^*$ for $|r_i| \neq 1$.

Let us suppose that the roots of the polynomial $H(z)$ are denoted by r_i as in (1).

We employ the following notation:

- $r_i = r_{j\text{in}}$ if the root r_i is inside the unit circle.
- $r_i = r_{j\text{out}}$ if the root r_i is outside the unit circle.
- $r_i = r_{j\text{o}}$ if the root r_i is on the unit circle.

Thus, we can write

$$H(z) = \left[\prod_j (z - r_{j\text{in}})\right] \left[\prod_j (z - r_{j\text{out}})\right] \left[\prod_j (z - r_{j\text{o}})\right]$$

or

$$H(z) = H_{\min}(z)H_{\max}(z)H_{\mathrm{o}}(z)$$

where $H_{\min}(z)$ is the MP part of $H(z)$ and $H_{\max}(z)$ is the maximum phase part of $H(z)$. The factor $H_o(z)$contains all the roots of $H(z)$ that lie on the unit circle.

In general, the frequency domain characteristics of a filter

$$H(\omega) = A(\omega)e^{\mathrm{j}\theta(\omega)}$$

are described in terms of both gain $A(\omega)$ and phase $\theta(\omega)$. The gain is always specified explicitly and approximated in any practical scheme, whereas the phase depends on its type, i.e., LP, MP or NMP. The group delay of a filter, $\tau_g(\omega)$, is defined as follows:

$$\tau_g = -\frac{d\theta(\omega)}{d\omega}.$$

For LP filters of order n it is easily shown that the group delay is

$$\tau_g(\theta) = n/2.$$

Note that for filters of very large order, the group delay will take very large value. For example for $n = 1024$ we have $\tau_g = 512$.

Some useful general points need to be made.

- For a range of applications with stringent specifications as in telecommunications, a typical FIR digital filter transfer function may be of order 200 or more. For such filters the group delay may be undesirable particularly when it approaches large values, when bidirectional human-to-human communication is not viable.
- Often in many applications the phase response is either unimportant or irrelevant. For example in some speech processing areas it is not significant. This form of freedom in the design of the filters is not normally taken into consideration by the existing FIR filter design methods.

At any rate, the design of MP FIR filters from LP FIR filters with specific amplitude requirements would inevitably lead to a stage of factorization in order to select the appropriate zeros and hence problems with imprecisions would arise.

We aim in this paper to derive the required nonlinear (minimum) phase FIR filter transfer functions from corresponding LP functions that are assumed to be designable by the Reméz exchange algorithm [5]. MATLAB provides a function called *Remez*, for the design of equiripple LP FIR filter transfer functions. The function requires as inputs the order of the filter n, the type of the filter and the size of each band. In this study, the LP filters used in the simulations are designed with the function *Remez*.

B. Given a LP Filter Design a MP Filter with the Same Amplitude Response: A Mathematical Description

In principle, to obtain a MP version of the given transfer function we can follow the steps below.

1) Either determine $H_{\max}(z)$ and reflect its zeros into the unit circle, or determine $H_{\min}(z)$ and make each of its zeros of multiplicity 2.
2) Find $H_{\mathrm{o}}(z)$.
3) Construct the transfer function as $T(z) = [H_{\min}(z)]^2 H_{\mathrm{o}}(z)$.

Then we shall have $|T(e^{\mathrm{j}\theta})| = |H(e^{\mathrm{j}\theta})|$.

Both step 1) and step 2) imply at first glance that a root finding procedure may be required. However, as already pointed out, root finding procedures are known to be inaccurate and unreliable for large order polynomials. Factorization without root finding forms also the basis of the procedure developed in [7], [8], and [17]. In [7] and [17] use is made of the real cepstral parameters, where the cepstral aliasing problem is recognized and careful procedures are recommended to reduce its effects. In [8] they approach the factorization problem from the Lagrange interpolation point of view. In the above procedures it is assumed that the zeros of the transfer function on the unit circle are *a priori* known. We make no such assumption in our present work.

An alternative and direct polynomial construction procedure without having to go through root estimation procedures is possible through the Root Moments of a given polynomial [25], [26], or the differential cepstrum [27], [28].

III. Root Moments

A. Definition of Root Moments

In relation to any polynomial $H(z) = \prod_{i=1}^{n}(z - r_i)$ defined as in (1), Newton defined a set of parameters that are functions of the roots of the related polynomial, given by

$$S_m = r_1^m + r_2^m + \cdots + r_n^m = \sum_{i=1}^{n} r_i^m \tag{2}$$

where r_i is the ith root of (1). In [25], [26], it is proven analytically that the roots of (1) are not needed explicitly to compute S_m in that these parameters can be determined directly from the coefficients h_i through an iterative procedure. The parameters S_m are known as the *root moments* of the polynomial $H(z)$. They are related to many signal processing operations, dominant amongst which is the differential cepstrum [28].

B. Iterative Estimation of Root Moments

By writing the polynomial (1) as a product of factors we can write $H'(z) = \sum_{i=1}^{n}(H(z)/(z-r_i))$ and given that $H(r_i) = 0$ we have

$$\begin{aligned} H'(z) = {} & nz^{n-1} + (S_1 + nh_1)z^{n-2} \\ & + (S_2 + h_1S_1 + nh_2)z^{n-3} + \cdots \\ & + (S_m + h_1S_{m-1} + h_2S_{m-2} + \cdots + nh_m) \\ & \cdot z^{n-m-1} + \cdots. \end{aligned}$$

By direct differentiation of (1) we have

$$\begin{aligned} H'(z) = {} & nz^{n-1} + (n-1)h_1z^{n-2} + (n-2)h_2z^{n-3} \\ & + \cdots + (n-m)h_mz^{n-m-1} + \cdots. \end{aligned} \tag{3}$$

Hence by equating the last two expressions we obtain the following fundamental relationships known as *Newton Identities*:

$$S_m + h_1S_{m-1} + h_2S_{m-2} + \cdots + mh_{m-n} = 0. \tag{4}$$

When the signal treated by this means is infinitely long, the above equation is repeatedly used to calculate successive values of the root moments. If the signal is of finite duration then for $m > n$

$$S_m + h_1 S_{m-1} + h_2 S_{m-2} + \cdots + h_n S_{m-n} = 0.$$

The same relationship as above can be used to calculate S_m for $m < 0$ by inserting successively values for m equal to $n-1, n-2, n-3, \ldots$ etc. It should be noted that S_m for either positive or negative values of m are evaluated recursively from the coefficients of (1) alone.

The above relationships also follow from the definition of the differential cepstrum and are essentially included in [28]. However in [28] n is assumed to be finite *a priori* known. This is only a minor point as the iteration in (4) do not require n to be finite and, hence, they can be applied to infinite duration signals. It is sufficient at this juncture to observe that both finite duration signals and infinite duration signals of exponential entire function type interpretation can be treated in the same way [25]. To facilitate the exposition, the parameters in (2) are referred to as the root moments. This terminology emphasizes the deviation from the differential cepstrum.

Essentially one can interpret the set of equations (4) as a transformation of the coefficients $\{h_r\}$ to the parameter set $\{S_m\}$ of the same cardinality. The transformations are one-to-one and hence we can have the following existence corollaries.

Corollary 1: Given a set of coefficients $\{h_r\}$ $r = 1, \ldots, n$, of the nth degree polynomial in (1), which has roots $\{r_i\}$ $i = 1, \ldots, n$, there exists a set of parameters $\{S_m\}$ $m = 1, \ldots, n$, $S_0 = n$, given by (2).

Corollary 2: Conversely given a set of root moments $\{S_m\}$ $m = 1, \ldots, n$, $S_0 = n$ there exists a set of coefficients $\{h_r\}$ $r = 1, \ldots, n$, for a polynomial as in (1) determinable recursively through (4). The proofs are self evident from the above analysis.

In our main problem we need the following result. Assume that the root moments of the polynomial $H_1(z)$ are $S_m^{H_1(z)}$ and the root moments of the polynomial $H_2(z)$ are $S_m^{H_2(z)}$. Then the root moments of the product $H(z) = H_1(z)H_2(z)$ are $S_m^{H(z)} = S_m^{H_1(z)} + S_m^{H_2(z)}$.

C. Noniterative Estimation of Root Moments

The Newton Identities yield the root moments of the entire polynomial, encompassing all the roots that lie within the complex plane. However, it is often the case, that a specific factor of a given polynomial $H(z)$ is required, such as the MP or the maximum phase factor. In this case its root moments can be determined in a different manner, by using the Cauchy Residue Theorem.

Let a closed contour Γ defined as $z = \rho(\theta)e^{\mathrm{j}\theta}$. It is assumed that we have no zeros on Γ.

Then it follows from the Cauchy residue theorem that the root moments of the factor of the polynomial whose roots lie within Γ are given by

$$I_\Gamma(m) = S_m^\Gamma = \frac{1}{2\pi \mathrm{j}} \oint_\Gamma \frac{H'(z)}{H(z)} z^m \, dz. \tag{5}$$

This is evident from the fact

$$S_m^\Gamma = \frac{1}{2\pi \mathrm{j}} \oint_\Gamma \sum_i \frac{1}{(z - r_i)} z^m \, dz$$

and the contribution to the integration are those due to those roots that lie within Γ.

We assume that the contour Γ can be described using an analytical expression.

In practice the contour integration will have to be effected directly from the coefficients of $H(z)$ and this can be done quite conveniently through the use of the DFT as it is shown below.

Equation (5) becomes for $z = \rho(\theta)e^{\mathrm{j}\theta}$

$$S_m^\Gamma = \frac{1}{2\pi \mathrm{j}} \int_{-\pi}^{\pi} g(\theta) e^{\mathrm{j}(m+1)\theta} \, d\theta \tag{6}$$

where

$$g(\theta) = \frac{H'(\rho(\theta)e^{\mathrm{j}\theta})}{H(\rho(\theta)e^{\mathrm{j}\theta})} \left(\frac{d\rho(\theta)}{d\theta} + \mathrm{j}\rho(\theta) \right) \rho^m(\theta). \tag{7}$$

Discretization of (6) suitable for DFT use requires values $\theta_k = (2\pi/N)k$, $k = 0, 1, \ldots, N-1$ for an N-point transform. Note that the value of N has to be large enough in order to approximate sufficiently the integral by a summation and this is an issue for further investigation. Therefore, we have the inverse DFT

$$S_m^\Gamma \approx \frac{1}{\mathrm{j}N} \sum_{k=0}^{N-1} g(\theta_k) e^{\mathrm{j}(m+1)\theta_k}. \tag{8}$$

Important Observation: If the contour of integration is the unit circle $|z| = 1$ then the resulting root moments from the above, correspond to those of the MP component of $H(z)$. In this case (8) is reduced to the form

$$S_m^{H_{\min}(z)} \approx \frac{1}{N} \sum_{k=0}^{N-1} \frac{H'(\theta_k)}{H(\theta_k)} e^{\mathrm{j}(m+1)\theta_k}. \tag{9}$$

To compute either (8) or (9), it is observed that on $z = \rho(\theta)e^{\mathrm{j}\theta}$ we can write

$$H'(\rho(\theta)e^{\mathrm{j}\theta}) = e^{\mathrm{j}(n-1)\theta} \sum_{i=0}^{n-1} (n-i) h_i \rho^{n-i-1}(\theta) e^{-\mathrm{j}i\theta}$$

which for $\theta = \theta_k$ can be computed as

$$H'\left(\rho(\theta_k)e^{\mathrm{j}\theta_k}\right) = e^{\mathrm{j}(n-1)\theta_k} \mathrm{DFT}\{(n-i)h_i\rho^{n-i-1}(\theta_k)\}. \tag{10}$$

Similarly we have

$$H(\rho(\theta_k)e^{\mathrm{j}\theta_k}) = e^{\mathrm{j}n\theta_k} \mathrm{DFT}\{h_i \rho^{n-i}(\theta_k)\} \tag{11}$$

and hence

$$g(\theta_k) = e^{-\mathrm{j}\theta_k} \frac{\mathrm{DFT}\{(n-i)h_i\rho^{n-i-1}(\theta_k)\}}{\mathrm{DFT}\{h_i\rho^{n-i}(\theta_k)\}} \cdot \left(\frac{d\rho(\theta)}{d\theta} + \mathrm{j}\rho(\theta) \right) \rho^m(\theta).$$

With N a power of 2 we can use the FFT algorithm.

The error in $S_0^{H_{\min}(z)} = n_{\rm in}$ due to the approximation of the integral by a summation, for two zeros located at $\rho e^{\pm j\phi}$, $0 \le \rho < 1$, is proven [29] to be equal to

$$\varepsilon_2 = 2 - 2\,\frac{1}{1 + \rho^{2^M}\left(\dfrac{\rho^{2^M} - \cos(2^M\phi)}{1 - \rho^{2^M}\cos(2^M\phi)}\right)}$$

and hence, the error due to an individual zero will be $\varepsilon_1 = \varepsilon_2/2$.

As easily shown the error of the approximation tends to zero as the number of points on the unit circle increase significantly. When we approximate the above integral by a summation what we obtain in practice is a real number which for $m = 0$ we need to round to the nearest integer. For this reason the total error, i.e., the error which is due to the total number of zeros inside the unit circle, should not be allowed to have an absolute value larger than 0.5, otherwise the rounding operation would give a false result. If the number of zeros inside the unit circle is $n_{\rm in}$, then the maximum acceptable error $\varepsilon_{\max}$ due to the contribution in the summation of each one of the zeros within the circle can be found, therefore, from the following inequality:

$$n_{\rm in}\varepsilon_{\max} < 0.5 \text{ or equivalently } \varepsilon_{\max} < \frac{0.5}{n_{\rm in}}.$$

D. Relationship Between the Coefficients of the Differential Cepstrum and the Root Moments

The differential cepstrum was first proposed by Polydoros and Fam [28].

For the coefficients of the differential cepstrum of $H(z)$ the notation $d_H(m)$ is used, where

$$d_H(m) = \frac{1}{2\pi j}\oint_{|z|=1} \frac{H'(z)}{H(z)}\, z^{m-1}\, dz$$

for any $m \ne 1$. Moreover for $m = 1$ they do not define $d_H(1)$ properly. Indeed they arbitrarily assign to it an overall delay factor which may be associated with the given function $H(z)$.

It is evident for our approach that

$$d_H(m) = S_m^{H_{\min}(z)}, \qquad m \ge 2$$
$$d_H(m) = -S_{-m}^{H_{\max}(z)}, \qquad m \le 0.$$

Moreover it follows that $d_H(1)$ should be defined as $d_H(1) = n_{\rm in}$ and not as given in [28].

In this form a clear distinction is apparent between the maximum phase and MP components of $d_H(m)$.

After completing the theoretical background used in this section we proceed to describe our approach to the problem.

IV. The New Design Algorithm

The algorithm relies on the direct and accurate extraction of the appropriate factors from the FIR LP transfer function $H(z)$ needed to implement $T(z)$ described above. The FIR LP transfer function is designed using the Reméz or any other similar existing algorithm according to the specifications of the user. The designed filter has roots on the unit circle and hence we cannot obtain the MP part by integrating around the unit circle. The approach we take follows the steps below [30]–[31].

1) We integrate around a circle centered at the origin and of radius less than unity. With a careful choice of the contour radius, the integration gives the root moments $S_1(m) = S_{\rm in}(m)$ that correspond to that part of the original FIR transfer function which has its zeros inside the unit circle, namely the MP part.

   Obviously the radius of the contour is of crucial importance. We have to ensure that the selected circle includes all the roots of the MP part of the original polynomial.
2) Integrate around a circle centered at the origin and of radius greater than unity. Again the radius of the contour must be selected carefully.

   A good selection in step 1) yields a correspondingly good selection in step 2), as the radius chosen in step 2), is the reciprocal of the radius selected in step 1).

   The integration produces the parameters $S_2(m) = S_{\rm in}(m) + S_{\rm o}(m)$ where $S_{\rm o}(m)$ are the root moments of that factor of the original FIR digital filter transfer function which has its zeros on the unit circle.
3) The required transfer function has the root moments $S(m) = 2S_{\rm in}(m) + S_{\rm o}(m)$, obtained as $S(m) = S_1(m) + S_2(m)$.
4) From step 3) and from the Newton Identities we form the required MP FIR transfer function. The order of this is the same as the order of the original polynomial and equal to $S_1(0) + S_2(0)$.

V. Estimation of the Radii of the Integration

The radii of integration in the above algorithm must be chosen so as to enclose the appropriate zeros of the given FIR digital filter transfer function. Thus for $S_1(m)$ the radius of the integration contour r must be such that $1 > r > \max(|r_{\rm in}|)$, while for $S_2(m)$ the radius of the integration contour r must be chosen such that $1 < r < \min(|r_{\rm out}|)$.

For equiripple piecewise constant filters the required radii can be estimated as follows.

Let us remove the LP factor from the frequency response to yield only a real function. This function now we shift vertically halfway between its maximum and minimum values. Since the initial transfer function is equiripple the result of these operations will be a real function of equiripple modulus almost everywhere, as shown in Fig. 1.

The ripple variation remains unchanged, namely a normalized response will vary between $1 + \delta$ and $1 - \delta$ almost everywhere except in the transition band.

An approximate representation of this zero pattern almost everywhere, is given by

$$C(z) = (z^n - a^n)\left(z^n - \frac{1}{a^n}\right). \tag{12}$$

The above transfer function is equiripple, LP and its zeros are located on two circles controlled by the parameter a. The amplitude characteristic is equiripple between the values

$$C_{\max} = a^{2n} + 2a^n + 1 \quad \text{and} \quad C_{\min} = a^{2n} - 2a^n + 1.$$

The mean between the minimum and the maximum value is $(C_{\max} + C_{\min})/2 = a^{2n} + 1$. To calculate the ripple, the

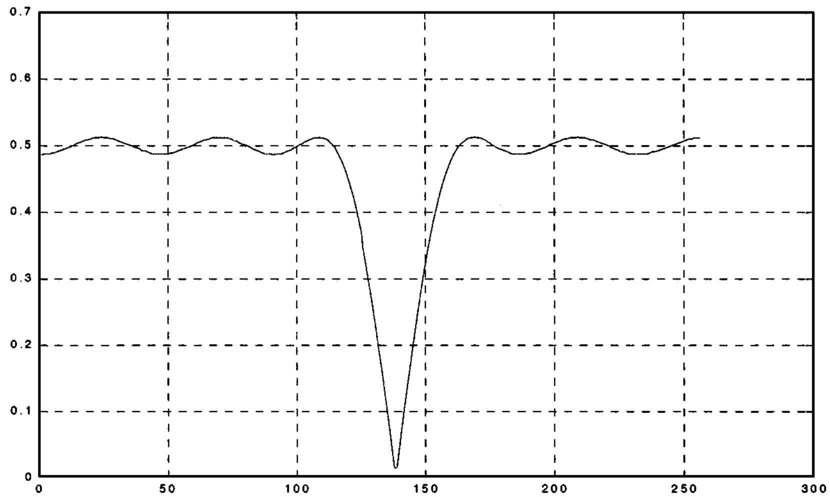

Fig. 1. Real function of equiripple modulus almost everywhere.

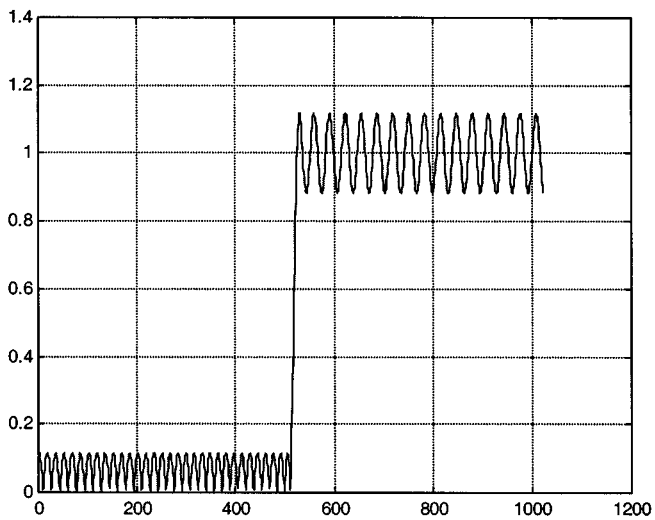

Fig. 2. Amplitude response of a high-pass filter of order 128, designed by the Reméz algorithm.

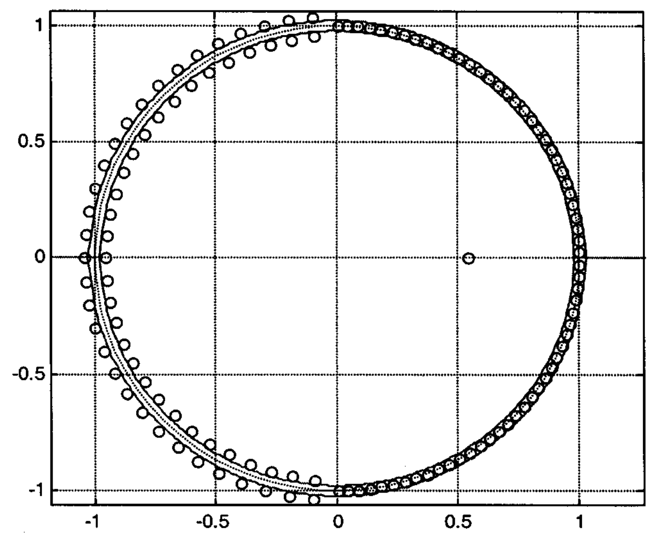

Fig. 3. Mixed-phase zeros on the complex plane and radii of integration.

amplitude response is normalized, by dividing with the mean $(a^{2n}+1)$, so that the maximum value now becomes

$$C_{\max} = \frac{a^{2n}+2a^n+1}{a^{2n}+1}.$$

We use the relationship

$$C_{\max} = \frac{a^{2n}+2a^n+1}{a^{2n}+1} = 1+\delta \Rightarrow \delta = \frac{2}{a^n+\dfrac{1}{a^n}}.$$

The quantity δ can be estimated from the initial transfer function created by the Reméz algorithm.

Hence we can estimate the radius of the circle on which the zeros are expected to be located as

$$a = \left(\frac{1}{\delta} \pm \sqrt{\left(\frac{1}{\delta^2}-1\right)}\right)^{1/n}.$$

For small ripple width the above can approximated to $a = (2/\delta)^{1/n}$.

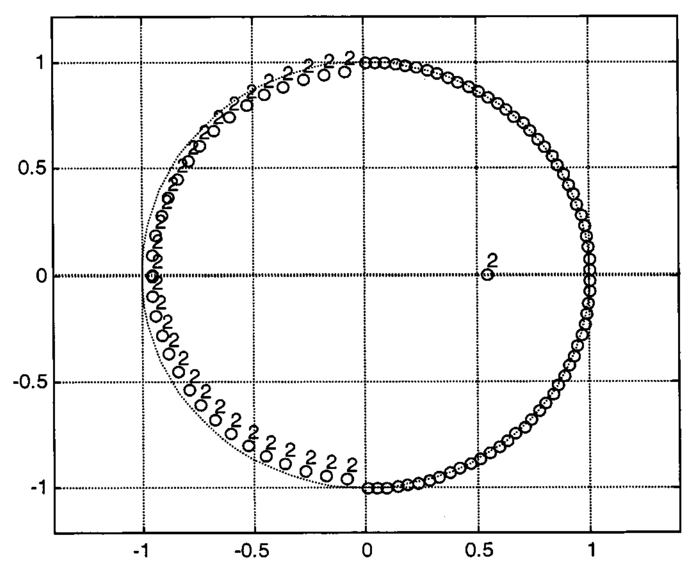

Fig. 4. Zeros of the reconstructed MP version filter.

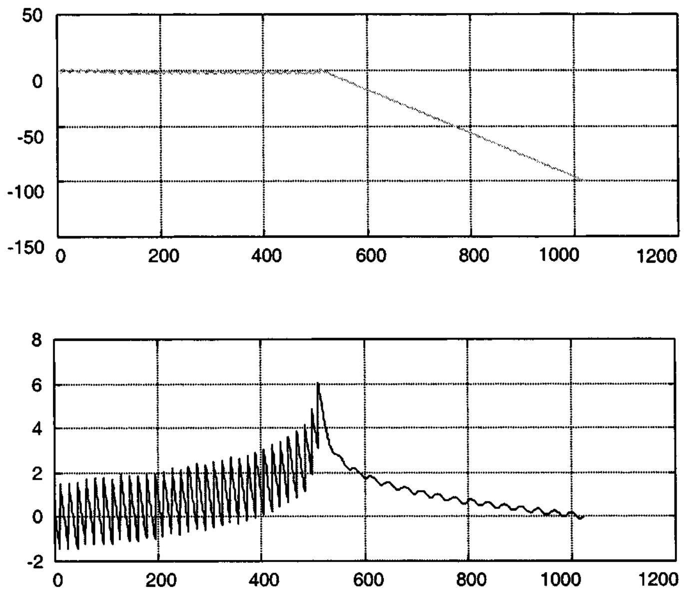

Fig. 5. Phase of the LP filter and of the reconstructed MP filter.

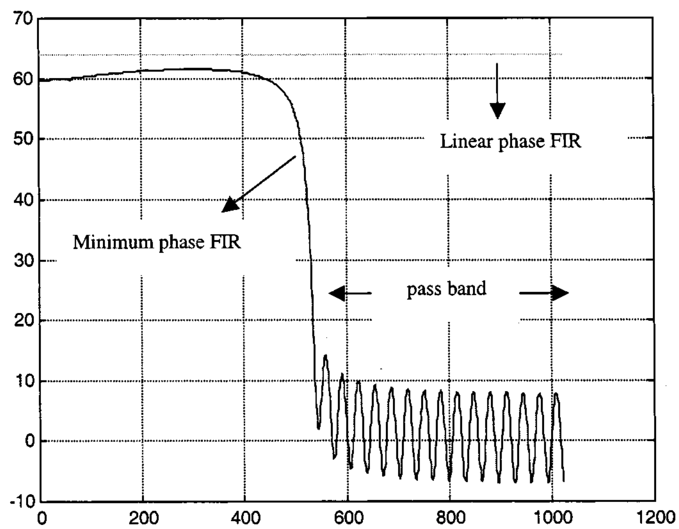

Fig. 6. Group delay of the LP filter and the reconstructed MP filter.

VI. Experimental Results

First we tested our algorithm using a LP high pass FIR filter of order 128. The filter was obtained by MATLAB's *Remez* function (Fig. 2). The sampling frequency is 1028 and the passband goes from 0.5π to π. After applying our algorithm (Fig. 3) we notice that the zeros outside the unit circle are inversed inside the unit circle (Fig. 4). The resulting MP filter has identical amplitude response to the one obtained using the *Remez* algorithm. Its phase is nonlinear (Fig. 5) but its group delay within the passband region is significantly lower compared to that of the LP filter (Fig. 6).

The algorithm was also applied to a LP low pass FIR filter of order 2048 (Fig. 7). The sampling frequency is 16 384 and the passband goes from 0 to 0.4π. Once more the amplitude responses of both linear and MP filters are identical. Similarly to the previous example the phase of the resulted MP filter is nonlinear (Fig. 8). Its group delay in the passband-is approximately 100 times lower than the one of the LP filter (Fig. 9).

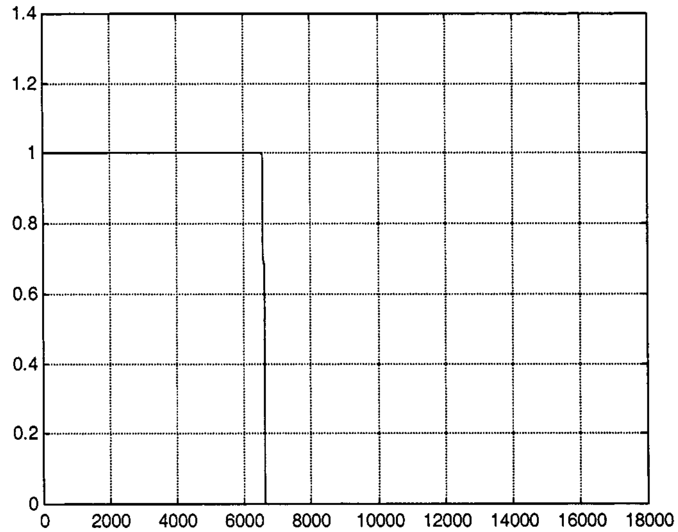

Fig. 7. Amplitude response of a low-pass filter of order 2048, designed by the Reméz algorithm.

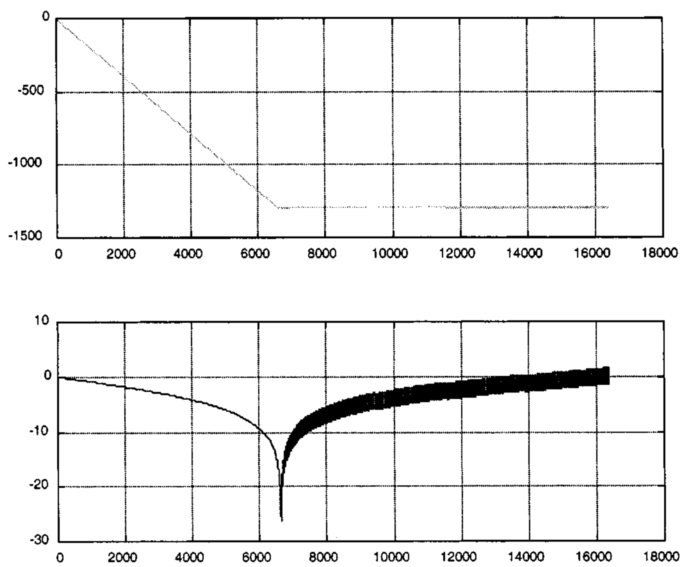

Fig. 8. Phase of the LP filter and of the reconstructed MP filter.

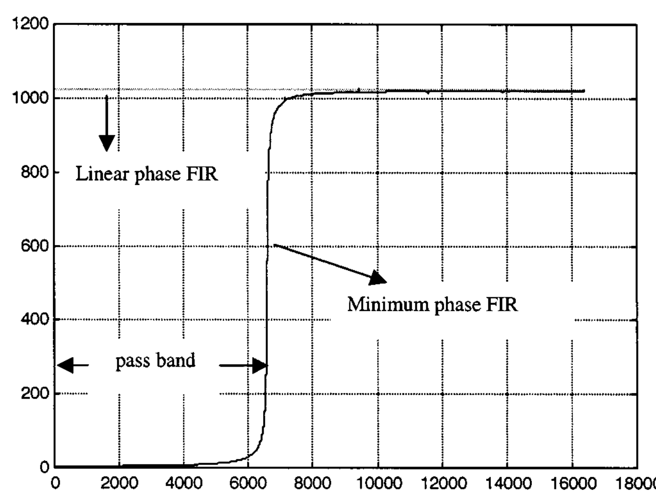

Fig. 9. Group delay of the LP filter and the reconstructed MP filter.

VII. Conclusions

In this paper we propose a novel approach to design a MP FIR digital filter transfer function from a given LP FIR transfer function which has identical amplitude response. The theoretical background of the proposed method relies on the root moments of a polynomial, a set of parameters that is described thoroughly throughout the paper. The algorithm presented in the paper is also described in an analytical form with sufficient theoretical background to enable the reader to follow the arguments. Appropriate proofs of assertions are also given wherever is necessary.

The LP systems obtained using the Reméz exchange algorithm. We deal with filters with equiripple amplitude response. We concentrate on very high degree polynomials (transfer functions) for which factorization procedures for root extraction are unreliable. More specifically, our methods allow orders larger than 2000 polynomial coefficients.

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