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Optimized robust control invariance for linear discrete-time systems: Theoretical foundations☆

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Abstract

This paper introduces the concept of optimized robust control invariance for discrete-time linear time-invariant systems subject to additive and bounded state disturbances. A novel characterization of two families of robust control invariant sets is given. The existence of a constraint admissible member of these families can be checked by solving a single and tractable convex programming problem in the generic linear-convex case and a standard linear/quadratic program when the constraints are polyhedral or polytopic. The solution of the same optimization problem yields the corresponding feedback control law that is, in general, set-valued. A procedure for selection of a point-valued, nonlinear control law is provided.

Keywords: Set invariance; Optimized robust control invariance; Constrained control; Robust control; Linear systems

1. Introduction

The theory of set invariance plays a fundamental role in control of constrained dynamical systems; see for instance the monograph (Aubin, 1991) and the survey paper (Blanchini, 1999). An important role for set invariance is evident in stability theory (La Salle, 1976). Set invariance, *inter alia*, provides useful tools for the synthesis of: (i) reference governors (Gilbert & Kolmanovsky, 1999), (ii) predictive controllers (Bemporad & Morari, 1999; Findeisen, Imsland, Allgöwer, & Foss, 2003; Mayne, 2001), (iii) robust time-optimal controllers (Bertsekas & Rhodes, 1971; Mayne & Schroeder, 1997) and (iv) robust, tube based, model predictive controllers (Mayne, Seron, & Raković, 2005; Raković, 2005). An application of set invariance in non-cooperative dynamic games is reported in Caravani and De Santis (2000, 2002) and Raković, De Santis, and Caravani (2005).

Given the importance of set invariance in control theory, the subject has been a topical research area over the last 40 years. A non-exhaustive list of the relevant references includes Blanchini (1994), Blanchini, Mesquine, and Miani (1995), Kolmanovsky and Gilbert (1998), Dórea and Hennet (1999), da Silva Jr. and Tarbouriech (1999), De Santis, Di Benedetto, and Berardi (2004), Raković, Kerrigan, Kouramas, and Mayne (2005) and Raković (2005). Most of these texts address computational issues and algorithmic procedures for the calculation of robust control and positively invariant sets as well as the application of these sets to robust control for constrained systems. One of the prime questions considered in the existing literature is the computation of the maximal robust control invariant (RCI) set (Aubin, 1991; Bertsekas, 1972; Blanchini, 1999; Dórea & Hennet, 1999). An algorithmically efficient technique is based on the computation of λ contractive sets (Blanchini, 1994). The theory and computation of minimal and maximal robust positively invariant (RPI) sets for the case of autonomous linear systems are examined in the important paper (Kolmanovsky &

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Gilbert, 1998). A method for finite time computation of an arbitrarily close outer RPI approximation of the minimal RPI set is discussed in Raković, Kerrigan, et al. (2005). Further contributions include the computation of maximal control and RCI sets for linear discrete-time systems (Dórea & Hennet, 1999) and stabilization of linear discrete-time systems subject to control constraints and an assigned initial condition set (Blanchini et al., 1995).

In this paper we provide a novel characterization of RCI sets. To this end, we introduce two families of RCI sets. The families are parameterized in a way that permits selection of RCI sets via tractable convex optimization problem in the generic linear-convex case. The techniques presented in this paper may be used to obtain improved RCI sets with respect to RPI sets that approximate the minimal RPI set (Raković, Kerrigan, et al., 2005) and, in principle, to obtain a RCI set that approximates the maximal robust control invariant set (see Remarks 1 and 3 in Section 3).

This paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 provides a novel characterization of two families of RCI sets. Section 4 discusses corresponding computational issues. Sections 5 presents some interesting numerical examples. Section 6 indicates potential extensions and provides concluding remarks.

Nomenclature and basic definitions: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$, $\mathbb{N}_{[q_1,q_2]} \triangleq \{q_1, q_1+1, \ldots, q_2-1, q_2\}$ for a given $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$ such that $q_1 < q_2$ and $\mathbb{N}_q$ denotes $\mathbb{N}_{[0,q]}$ for $q \in \mathbb{N}$. Let $\mathbb{B}_p^q(r) \triangleq \{x \in \mathbb{R}^q \mid |x|_p \leqslant r\}$ be a closed p-norm ball in $\mathbb{R}^q$, where $r \geqslant 0$ and $|\cdot|_p$ denotes the vector p-norm. Given two sets $\mathscr{A}$ and $\mathscr{B}$, such that $\mathscr{A} \subset \mathbb{R}^n$ and $\mathscr{B} \subset \mathbb{R}^n$, the Minkowski set addition is defined by $\mathscr{A} \oplus \mathscr{B} \triangleq \{a + b \mid a \in \mathscr{A},\ b \in \mathscr{B}\}$. Given a vector $v \in \mathbb{R}^n$ (which could be a sum of q vectors) and a set $\mathscr{S} \subset \mathbb{R}^n$, we write $v \oplus \mathscr{S}$ to denote $\{v\} \oplus \mathscr{S}$. Given the sequence of sets $\{\mathscr{S}_i \subset \mathbb{R}^n\}$, $i \in \mathbb{N}_{[a,b]}$ with $(a, b) \in \mathbb{N} \times \mathbb{N}$, $b > a$, we define $\bigoplus_{i=a}^{b} \mathscr{S}_i \triangleq \mathscr{S}_a \oplus \cdots \oplus \mathscr{S}_b$. Given a set $\mathscr{S} \subset \mathbb{R}^n$ and a real matrix M of compatible dimensions (which could be just a scalar) we define $M\mathscr{S} \triangleq \{Ms \mid s \in \mathscr{S}\}$. The controllability index of a given matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is denoted by $\mathscr{I}(A, B)$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces. A *polytope* is a closed and bounded polyhedron. A set $\mathscr{S} \subset \mathbb{R}^n$ is a C set if it is compact, convex and contains the origin. A C set $\mathscr{S}$ is proper if it has a non-empty interior.

2. Preliminaries

We consider the following discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + Bu + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action, x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown disturbance and $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. The disturbance w is persistent, but contained in a set $\mathbb{W} \subset \mathbb{R}^n$. In this paper we adopt the standing assumption:

Assumption 2.1. The matrix pair (A, B) is controllable and $\mathbb{W}$ is a C set.

The system (2.1) is subject to the following set of hard state and control constraints:

$$(x, u) \in \mathbb{X} \times \mathbb{U}, \tag{2.2}$$

where $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are, for simplicity, polyhedral and polytopic sets, respectively.

We first recall some well known definitions from set invariance theory (Blanchini, 1999):

Definition 2.1. A set $\Omega \subset \mathbb{R}^n$ is a *robust control invariant* (RCI) set for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and for all $x \in \Omega$ there exists a $u \in \mathbb{U}$ such that $Ax + Bu + w \in \Omega$ for all $w \in \mathbb{W}$.

If a set $\Omega \subset \mathbb{R}^n$ is RCI for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$, then any (point-valued) control law $\mu : \Omega \to \mathbb{U}$ satisfying

$$\mu(x) \in \{u \in \mathbb{U} \mid Ax + Bu \oplus \mathbb{W} \subseteq \Omega\}, \quad x \in \Omega, \tag{2.3}$$

ensures, by construction that $Ax + B\mu(x) \oplus \mathbb{W} \subseteq \Omega$ for all $x \in \Omega$. Given a control law $v : \mathbb{R}^n \to \mathbb{R}^m$, let

$$\mathbb{X}_v \triangleq \{x \in \mathbb{X} \mid v(x) \in \mathbb{U}\}. \tag{2.4}$$

Definition 2.2. A set $\Omega \subset \mathbb{R}^n$ is a *robust positively invariant* (RPI) set for the system $x^+ = Ax + Bv(x) + w$ and constraint set $(\mathbb{X}_v, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}_v$ and, for all $x \in \Omega$, $Ax + Bv(x) + w \in \Omega$ for all $w \in \mathbb{W}$.

In this paper, we exploit linearity and time invariance of Eq. (2.1), Assumption 2.1 and the contraction principle to parametrize RCI sets for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$ along with the associated control policy. The effect of the uncertainty is taken into account by means of the corresponding forward reach sets, which at time $k \in \mathbb{N}_+$ take the form $T_0\mathbb{W} \oplus T_1\mathbb{W} \oplus \cdots \oplus T_{k-1}\mathbb{W}$, where $T_i \in \mathbb{R}^{n\times n}$, $i \in \mathbb{N}_{k-1}$ with $T_0 = I$ and each T_i, $i \in \mathbb{N}_{[1,k-1]}$ is parametrized by the matrix pair (A, B) and a set of matrices $M_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{N}_{k-1}$. Assumption 2.1 permits the choice, for a suitable $k \in \mathbb{N}$, of the matrices $M_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{N}_{k-1}$, hence the matrices T_i, $i \in \mathbb{N}_{k-1}$ such that $T_k\mathbb{W} \subseteq \alpha\mathbb{W}$ for some $\alpha \in [0, 1)$. Consequently, for any integer k such that $T_k\mathbb{W} \subseteq \alpha\mathbb{W}$, $\alpha \in [0, 1)$, we can ensure that $T_{rk+l}\mathbb{W} \subseteq \alpha^r T_l\mathbb{W}$ for $r \in \mathbb{N}_+$, $l \in \mathbb{N}_{k-1}$ and $\bigoplus_{i=0}^{\infty} T_i\mathbb{W} \subseteq (1-\alpha)^{-1}\bigoplus_{i=0}^{k-1} T_i\mathbb{W}$ (for appropriate $T_i \in \mathbb{R}^{n\times n}$). We examine the parametrized sets $(1-\alpha)^{-1}\bigoplus_{i=0}^{k-1} T_i\mathbb{W}$ and establish that a suitable choice of the matrices $M_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{N}_{k-1}$ (hence the corresponding linear transformations T_i) ensures that these sets are RCI for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$. The fact that the sets are parametrized by matrices $M_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{N}_{k-1}$ (as well as the corresponding control policy) is utilized for the constrained case to obtain an efficient computational scheme for a selection of RCI sets (for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$) from the family of parametrized RCI sets.

This paper reports a novel characterization of two families of RCI sets. Existence of a constraint admissible member of these families (and the computation of the corresponding feedback controller) can be established by solving a *single convex programming problem* in the generic linear-convex case (a linear/quadratic programming (LP/QP) problem, when constraints on the states, controls and disturbances are polytopic or polyhedral, the system is linear and the objective function is linear/quadratic). To our best knowledge, the results reported here are novel and provide a generalization and detailed exposition of the results established in Raković (2005) and partially presented in Raković and Mayne (2005) and Raković, Mayne, Kerrigan, and Kouramas (2005).

3. Families of parameterized RCI sets

First, we introduce two families of RCI sets for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$, i.e., for the case when $\mathbb{X}=\mathbb{R}^n$ and $\mathbb{U}=\mathbb{R}^m$. For $k \in \mathbb{N}_+$ and $i \in \mathbb{N}_+$, let the matrices $\mathbf{M}_k \in \mathbb{R}^{km\times n}$ and $\mathscr{C}_i \in \mathbb{R}^{n\times km}$ be defined as

$$\mathbf{M}_k \triangleq [M_0^{\mathrm{T}} \;\; M_1^{\mathrm{T}} \;\; \cdots \;\; M_{k-2}^{\mathrm{T}} \;\; M_{k-1}^{T}]^{\mathrm{T}}, \tag{3.1}$$

$$\mathscr{C}_i \triangleq [A^{i-1}B \;\; A^{i-2}B \;\; \cdots \;\; AB \;\; B \;\; \mathbf{0} \;\; \cdots \;\; \mathbf{0}], \tag{3.2}$$

with $\mathscr{C}_0 \triangleq \mathbf{0}$ and each sub matrix $M_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{N}$. We consider the sets $R_k(\mathbf{M}_k)$, $k \in \mathbb{N}_+$ defined by

$$R_k(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} (A^i + \mathscr{C}_i \mathbf{M}_k)\mathbb{W}, \quad k \in \mathbb{N}_+. \tag{3.3}$$

Note that for any finite integer $k \in \mathbb{N}_+$ and any arbitrary, fixed, $\mathbf{M}_k$, $R_k(\mathbf{M}_k)$ is a C set, since it is the Minkowski sum of a finite number of C sets ($\mathbb{W}$ is a C set so is every summand $(A^j + \mathscr{C}_j \mathbf{M}_k)\mathbb{W}$, $j \in \mathbb{N}_{k-1}$).

3.1. Family of $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ RCI sets

A condition under which a simple scaling of the set $R_k(\mathbf{M}_k)$ defined by (3.3) is an RCI set for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$ is

$$(A^k + \mathscr{C}_k \mathbf{M}_k)\mathbb{W} \subseteq \alpha\mathbb{W}, \quad \alpha \in [0, 1). \tag{3.4}$$

Since the pair (A, B) is assumed to be controllable, a suitable choice of $\mathbf{M}_k$ and an α satisfying (3.4) certainly exists for any $k \geqslant \mathscr{I}(A, B)$. This follows from the fact that the set of linear equations $(A^k + \mathscr{C}_k \mathbf{M}_k) = \alpha I$ in $\mathbf{M}_k$ has at least one, and generally more than one, solution for any $\alpha \in [0, 1)$ and $k \geqslant \mathscr{I}(A, B)$. For all $\mathbf{M}_k$ and α such that $(A^k + \mathscr{C}_k \mathbf{M}_k) = \alpha I, \alpha \in [0, 1)$, the set inclusion (3.4) is certainly satisfied for any C set $\mathbb{W}$. When $\mathbb{W}$ is a proper C set the condition (3.4) is easier to satisfy, since $(A^k + \mathscr{C}_k \mathbf{M}_k) = \alpha I$ implies $(A^k + \mathscr{C}_k \mathbf{M}_k)\mathbb{W} \subseteq \alpha\mathbb{W}$, while $(A^k + \mathscr{C}_k \mathbf{M}_k)\mathbb{W} \subseteq \alpha\mathbb{W}$ does not necessarily imply that $(A^k + \mathscr{C}_k \mathbf{M}_k) = \alpha I$.

Let $\mathbb{M}_{(k,\alpha)}$ be defined by

$$\mathbb{M}_{(k,\alpha)} \triangleq \{(\mathbf{M}_k, \alpha) \in \mathbb{R}^{km\times n} \times \mathbb{R} \mid \alpha \in [0, 1), \\ (A^k + \mathscr{C}_k \mathbf{M}_k)\mathbb{W} \subseteq \alpha\mathbb{W}\}. \tag{3.5}$$

We consider the family of $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ sets, where

$$R_{(k,\alpha)}(\mathbf{M}_k, \alpha) \triangleq (1-\alpha)^{-1} R_k(\mathbf{M}_k) \tag{3.6}$$

and establish the following claim, a proof of which is given in Appendix A.1.

Theorem 3.1. *Suppose Assumption* 2.1 *holds. Then*: (i) *the set* $\mathbb{M}_{(k,\alpha)}$ *defined by* (3.5) *is non-empty for every integer* $k \geqslant \mathscr{I}(A, B)$ *and* (ii) *given any integer* $k \in \mathbb{N}$ *such that* $\mathbb{M}_{(k,\alpha)} \neq \emptyset$, *any* $(\mathbf{M}_k, \alpha) \in \mathbb{M}_{(k,\alpha)}$ *and the corresponding set* $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ *defined by* (3.6), *there exists a control law* $v : R_{(k,\alpha)}(\mathbf{M}_k, \alpha) \to \mathbb{R}^m$ *such that* $Ax + Bv(x) \oplus \mathbb{W} \subseteq R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ $\forall x \in R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$.

3.2. Family of $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ RCI sets

The family of sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ is merely a subset of a richer family of RCI sets that we identify next. Let the set $\mathscr{Z}$ denote the set of equilibrium points for the nominal part of (2.1):

$$\mathscr{Z} \triangleq \{(\bar{x}, \bar{u}) \mid (A - I)\bar{x} + B\bar{u} = \mathbf{0}\}, \tag{3.7}$$

and let also

$$\mathscr{X} \triangleq \{\bar{x} \mid \exists\, \bar{u} \text{ such that } (\bar{x}, \bar{u}) \in \mathscr{Z}\} \quad \text{and} \tag{3.8}$$

$$\mathbb{U}(\bar{x}) \triangleq \{\bar{u} \mid (\bar{x}, \bar{u}) \in \mathscr{Z}\}, \quad \bar{x} \in \mathscr{X}. \tag{3.9}$$

We recall the following result employed in Raković and Mayne (2005) and Raković (2005), a proof of which is given in Appendix A.2.

Proposition 3.1. *For any* $\bar{x} \in \mathscr{X}$, *where* $\mathscr{X}$ *is defined by* (3.8), *and any RCI set* Ω *for system* (2.1) *and constraint set* $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$, $\Omega_{\bar{x}} \triangleq \bar{x} \oplus \Omega$ *is a RCI set for system* (2.1) *and constraint set* $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$.

Consider the family of $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ set, where

$$S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \triangleq \bar{x} \oplus R_{(k,\alpha)}(\mathbf{M}_k, \alpha), \tag{3.10}$$

for $k \in \mathbb{N}_+$ and $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ is defined by (3.6). Let $\mathbb{L}_{(k,\alpha)}$ be defined by

$$\mathbb{L}_{(k,\alpha)} \triangleq \{(\bar{x}, \mathbf{M}_k, \alpha) \in \mathbb{R}^n \times \mathbb{R}^{km\times n} \times \mathbb{R} \mid \bar{x} \in \mathscr{X}, \\ (\mathbf{M}_k, \alpha) \in \mathbb{M}_{(k,\alpha)}\}, \tag{3.11}$$

where $\mathbb{M}_{(k,\alpha)}$ and $\mathscr{X}$ are defined in (3.5) and (3.8). Combining Theorem 3.1 and Proposition 3.1 we have our next main result:

Theorem 3.2. *Suppose Assumption* 2.1 *holds. Then*: (i) *the set* $\mathbb{L}_{(k,\alpha)}$ *defined by* (3.11) *is non-empty for every integer* $k \geqslant \mathscr{I}(A, B)$ *and* (ii) *given any integer* $k \in \mathbb{N}$ *such that* $\mathbb{L}_{(k,\alpha)} \neq \emptyset$, *any* $(\bar{x}, \mathbf{M}_k, \alpha) \in \mathbb{L}_{(k,\alpha)}$ *and the corresponding set* $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ *defined by* (3.10) *there exists a control law* $\mu : S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \to \mathbb{R}^m$ *such that* $Ax + B\mu(x) \oplus \mathbb{W} \subseteq S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ $\forall x \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$.

An important consequence of Theorems 3.1 and 3.2 is that, for a suitable $k \in \mathbb{N}$ (in fact, for any $k \geqslant \mathscr{I}(A, B)$), the sets

$R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ are, by construction, RCI for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$. Note that, the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ are the Minkowski sums of a finite number of C sets, that is they are *finitely determined by k*, and are relatively easily computed and/or represented if, in addition, $\mathbb{W}$ is a polytope. The family of $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ sets is parametrized by the matrix-scalar pair $(\mathbf{M}_k, \alpha)$ while the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ is parametrized by the pair $(\bar{x}, \bar{u})$ and the pair $(\mathbf{M}_k, \alpha)$.

Remark 1. The shape of RCI sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ can be improved by minimizing a generalized Minkowski (polytopic) norm of these sets, as proposed in Raković, Mayne, et al. (2005), Raković and Mayne (2005) and Raković (2005). The necessary computational details can be obtained from the results given in Section 4. Additionally, it is straight-forward to demonstrate that a family of RPI sets reported in Raković, Kerrigan, et al. (2005) (for the case when an arbitrary stabilizing state feedback $u = Kx$ is applied to the system (2.1) and $\mathbb{W}$ is a proper C set) is a subset of the family of $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ set. It is also possible to consider the RCI sets $\tilde{R}_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $\tilde{S}_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ defined, for any C set $\tilde{\mathbb{W}}$ in $\mathbb{R}^n$ such that $\mathbb{W} \subseteq \tilde{\mathbb{W}}$, in the same way as the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ with $\mathbb{W}$ replaced by $\tilde{\mathbb{W}}$.

Remark 2. Since $R_{(k,\alpha)}(\mathbf{M}_k, \alpha) = S_{(k,\alpha)}(\mathbf{0}, \mathbf{M}_k, \alpha)$ by (3.10) and in view of Theorems 3.1 and 3.2 and Proposition 3.1, we provide analysis in the sequel of this paper for the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$. The results are directly applicable for the set $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ with a suitable modification.

3.3. Invariance inducing feedback control laws

An invariance inducing feedback control law $\mu(\cdot)$ of Theorem 3.2 can be obtained by using (2.3). However, an interesting feature of set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ is the fact that we are able to characterize explicitly (in terms of the pair $(\bar{x}, \bar{u})$ and the pair $(\mathbf{M}_k, \alpha)$) a set-valued control law that induces, by construction, robust control invariance of set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$.

For any $k \in \mathbb{N}_+$ and corresponding $\mathbf{M}_k$, let the matrices $\mathscr{M}_k \in \mathbb{R}^{m\times kn}$ and $\mathscr{D}_k \in \mathbb{R}^{n\times kn}$ be defined by

$$\mathscr{M}_k \triangleq [M_{k-1} \;\; M_{k-2} \;\; \cdots \;\; M_1 \;\; M_0], \tag{3.12}$$

$$\mathscr{D}_k \triangleq [A^{k-1} + \mathscr{C}_{k-1}\mathbf{M}_k \;\; \cdots \;\; A + \mathscr{C}_1\mathbf{M}_k \;\; I]. \tag{3.13}$$

For any integer $k \in \mathbb{N}$ such that $\mathbb{L}_{(k,\alpha)} \neq \emptyset$ and for any $(\bar{x}, \mathbf{M}_k, \alpha) \in \mathbb{L}_{(k,\alpha)}$ the feedback control law $\mu : S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \to \mathbb{R}^m$ in Theorem 3.2 can be obtained as a selection from the set-valued map defined, for $x \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$, by

$$\mathscr{U}(x) \triangleq \{\bar{u} + \mathscr{M}_k\mathbf{w} \mid \bar{u} \in \mathbb{U}(\bar{x}),\ \mathbf{w} \in \mathbf{W}_\alpha(x)\}, \tag{3.14}$$

where $\mathbb{U}(\bar{x})$ is defined in (3.9), $\mathscr{M}_k$ is given by (3.12) and the set-valued map $\mathbf{W}_\alpha(x)$ is defined for each $x \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ by

$$\mathbf{W}_\alpha(x) \triangleq \{\mathbf{w} \mid \mathbf{w} \in (1-\alpha)^{-1}\mathbf{W}^k,\ \bar{x} + \mathscr{D}_k\mathbf{w} = x\}, \tag{3.15}$$

where $\mathbf{w} \triangleq \{w_0, w_1, \ldots, w_{k-1}\}$ (with its appropriate vectorized form $\mathbf{w} = [w_0^{\mathrm{T}} \;\; w_1^{\mathrm{T}} \;\; \cdots \;\; w_{k-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{kn}$ used in algebraic expressions), $\mathbf{W}^k \triangleq \mathbb{W} \times \mathbb{W} \times \cdots \times \mathbb{W}$ and $\mathscr{D}_k$ is given by (3.13). A point-valued control law $\mu(x) \in \mathscr{U}(x)$ satisfying Theorem 3.2 can be defined as

$$\mu(x) \triangleq \bar{u} + \mathscr{M}_k\mathbf{w}(x), \quad \mathbf{w}(x) \in \mathbf{W}_\alpha(x), \tag{3.16}$$

while a suitable selection of $\mathbf{w}(x) \in \mathbf{W}_\alpha(x)$ is

$$\mathbf{w}^0(x) \triangleq \arg\min_{\mathbf{w}}\{|\mathbf{w}|_2^2 \mid \mathbf{w} \in \mathbf{W}_\alpha(x)\}, \tag{3.17}$$

where $\mathbf{W}_\alpha(x)$ is defined by (3.15) and $\bar{u} \in \mathbb{U}(\bar{x})$ with $\mathbb{U}(\bar{x})$ defined in (3.9). Note that any $\bar{u} \in \mathbb{U}(\bar{x})$ can be used in (3.16) for the unconstrained case, while it is necessary to restrict choices of $\bar{u} \in \mathbb{U}(\bar{x})$ in the constrained case. If $\mathbb{W}$ is a polytopic set, then the function $\mathbf{w}^0(\cdot)$ is in general piecewise affine though in certain degenerate cases $\mathbf{w}^0(\cdot)$ might be linear or piecewise linear, being the solution of a *parametric* quadratic program (see, e.g., Bemporad, Morari, Dua, & Pistikopoulos, 2002). If the function $\mathbf{w}^0(\cdot)$ is used in (3.16), the feedback control law $\mu : S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \to \mathbb{R}^m$ is piecewise affine since it is an affine map of a piecewise affine function.

The fact that $\mu(x) \triangleq \bar{u} + \mathscr{M}_k\mathbf{w}(x)$, for any $\mathbf{w}(x) \in \mathbf{W}_\alpha(x)$, induces robust control invariance of the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ allows for extra flexibility in generating appropriate control action. For instance, given a state $x \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ and $\mathbf{w}(x) = \{w_0(x), w_1(x), \ldots, w_{k-1}(x)\} \in \mathbf{W}_\alpha(x)$ an arbitrary selection of $\mathbf{w}(x^+) \in \mathbf{W}_\alpha(x^+)$ for any successor state $x^+ \in Ax + B\mu(x) \oplus \mathbb{W}$ can be obtained by a direct and simple algebraic calculation. Given a measured successor state $x^+ \in Ax + B\mu(x) \oplus \mathbb{W}$ we can define $w(x^+) = x^+ - (Ax + B\mu(x))$ and $\mathbf{w}(x^+) \triangleq \{w_1(x), \ldots, w_{k-1}(x), w(x^+)\}$. Direct verification yields that $w(x^+) \in \mathbb{W} \subseteq (1-\alpha)^{-1}\mathbb{W}$ and $\mathbf{w}(x^+) \in \mathbf{W}_\alpha(x^+)$. Therefore, given an $x(0) \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ the function $\mathbf{w}^0(x(0))$ (3.17)can be used at the initiation of the controller and $\mathbf{w}(x(i)), i \in \mathbb{N}$ can be generated by simple algebraic manipulations at future time instances permitting a computationally simple invariance inducing controller implementation.

3.4. RCI set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$

We proceed to show how to check the existence of a set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ that is RCI for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$. Henceforth, we assume the following:

Assumption 3.1. The set $\mathscr{Z}$ defined in (3.7) is such that $\mathscr{Z} \cap (\textit{interior}(\mathbb{X}) \times \textit{interior}(\mathbb{U})) \neq \emptyset$.

In view of the parametrization of the control law $\mu(\cdot)$ (see (3.15) and (3.16)), we define

$$U(\bar{u}, \mathbf{M}_k, \alpha) \triangleq \bar{u} \oplus (1-\alpha)^{-1}\bigoplus_{i=0}^{k-1} M_i\mathbb{W}. \tag{3.18}$$

Clearly, the state and control constraints are satisfied if

$$S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \subseteq \mathbb{X} \quad \text{and} \quad U(\bar{u}, \mathbf{M}_k, \alpha) \subseteq \mathbb{U}, \tag{3.19}$$

where $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ and $U(\bar{u}, \mathbf{M}_k, \alpha)$ are defined by (3.10) and (3.18), respectively. For any $k \in \mathbb{N}_+$, let $\theta_k \triangleq (\bar{x}, \bar{u}, \mathbf{M}_k, \alpha)$ and let the set Θ_k be defined by

$$\Theta_k \triangleq \{\theta_k \mid (\bar{x}, \bar{u}, \mathbf{M}_k, \alpha) \in \mathscr{Z} \times \mathbb{M}_{(k,\alpha)}, \\ S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \subseteq \mathbb{X},\ U(\bar{u}, \mathbf{M}_k, \alpha) \subseteq \mathbb{U}\}, \tag{3.20}$$

where we recall that $\mathbb{M}_{(k,\alpha)}$ and $\mathscr{Z}$ are defined in (3.5) and (3.7). If, for a given $k \in \mathbb{N}_+$, the set Θ_k is a non-empty set then it follows by Theorem 3.2 and the definition of the set Θ_k, that the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$, for any $\theta_k = (\bar{x}, \bar{u}, \mathbf{M}_k, \alpha) \in \Theta_k$, is RCI for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$. Let

$$\mathscr{K} \triangleq \{k \in \mathbb{N}_+ \mid \Theta_k \neq \emptyset\}, \tag{3.21}$$

where Θ_k is defined in (3.20).

Theorem 3.3. *Suppose $\mathscr{K} \neq \emptyset$, where $\mathscr{K}$ is defined in* (3.21). *Then, given any $k \in \mathscr{K}$, any $\theta_k \in \Theta_k$ and the corresponding set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ defined by* (3.10): (i) *$S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \subseteq \mathbb{X}$ and* (ii) *there exists a control law $\mu : S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \to \mathbb{U}$ such that $Ax + B\mu(x) \oplus \mathbb{W} \subseteq S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)\ \forall x \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$.*

Additional structure of the set Θ_k is established by:

Proposition 3.2. *Suppose $\mathscr{K} \neq \emptyset$, where $\mathscr{K}$ is defined in* (3.21). *Then*: (i) *the set Θ_k is convex* (*if not empty*) *in* $(\bar{x}, \bar{u}, \mathbf{M}_k)$ *for any fixed $\alpha \in [0, 1)$ and* (ii) *if, in addition, the set Θ_k is non-empty for some $k \in \mathbb{N}$ with $\alpha = 0$, the sets Θ_s are non-empty for all $s \in \mathbb{N},\ s \geqslant k$.*

The proof of Proposition 3.2 is provided in Appendix A.3.

Remark 3. We conclude this section by exploiting the facts that the union of any number of RCI sets is RCI and the convex hull of a RCI set is also a RCI set when the respective state and control constraint sets ($\mathbb{X}$ and $\mathbb{U}$) are convex. For every $k \in \mathscr{K}$, the variable $\theta_k \in \Theta_k$ yields by Theorem 3.3 a RCI set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ as well as a RCI set $\mathscr{S}_{\Theta_k}$ (for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$) defined, for $k \in \mathscr{K}$ (see (3.21)), by

$$\mathscr{S}_{\Theta_k} \triangleq \bigcup_{(\bar{x}, \bar{u}, \mathbf{M}_k, \alpha) \in \Theta_k} S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha). \tag{3.22}$$

Hence, the sets $\mathscr{S}_{\mathscr{K}}$ and $\mathscr{S}^C_{\mathscr{K}}$ defined by

$$\mathscr{S}_{\mathscr{K}} \triangleq \bigcup_{k \in \mathscr{K}} \mathscr{S}_{\Theta_k} \quad \text{and} \quad \mathscr{S}^C_{\mathscr{K}} \triangleq \operatorname{convh}(\mathscr{S}_{\mathscr{K}}), \tag{3.23}$$

where $\operatorname{convh}(\Omega)$ stands for the closed convex hull of a set Ω, are also RCI sets for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$. The RCI set $\mathscr{S}^C_{\mathscr{K}}$ can, in principle, be used to obtain inner approximations of the maximal RCI set (a RCI set that contains all RCI sets) for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$. We provide an example in Section 5 to illustrate that it is not necessarily true that the set $\mathscr{S}^C_{\mathscr{K}}$ is equal to the maximal RCI set for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$.

4. Computational aspects

In order to exploit the results of Theorems 3.1–3.3 and Proposition 3.2 and extract a RCI set for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ we consider the following optimization problem defined for $k \in \mathbb{N}$:

$$\mathbb{P}_k :\ \theta_k^0 \in \arg\inf_{\theta_k} \{f(\theta_k) \mid \theta_k \in \Theta_k\}, \tag{4.1}$$

where the constraint set Θ_k is given in (3.20) and the objective function $f(\cdot)$ is preferably chosen to be convex and to provide a suitable criterion for selection of the RCI set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$. In view of Proposition 3.2, we are able to obtain a convex programming problem if the pair $(k, \alpha) \in \mathbb{N} \times [0, 1)$ is fixed and the objective function $f(\cdot)$ is convex.

4.1. Linear-polytopic case: efficient computations

Before proceeding, we need the following result, which follows from basic properties of support functions of sets and duality results in convex analysis (see, e.g., Rockafellar, 1970; Schneider, 1993):

Proposition 4.1. *Given two non-empty polyhedra $X \triangleq \{x \in \mathbb{R}^n | Fx \leqslant g\}$ and $Y \triangleq \{y \in \mathbb{R}^p | Qy \leqslant s\}$ and a set of matrices $\{L_0, L_1, \ldots, L_{k-1}\}$, where $L_i \in \mathbb{R}^{p \times n}$ for $i = 0, 1, \ldots, k-1$, the set inclusion*

$$\bigoplus_{i=0}^{k-1} L_i X \subseteq Y \tag{4.2}$$

is true if and only if there exists a set of non-negative matrices $\{Z_0, Z_1, \ldots, Z_{k-1}\}$ such that

$$\sum_{i=0}^{k-1} Z_i g \leqslant s \quad \text{and} \quad Z_i F = Q L_i \ \ \forall i \in \mathbb{N}_{k-1}. \tag{4.3}$$

Remark 4. Note that in the special case when $k = 1$ and $L_0 = I$ for determining whether $X \subseteq Y$, the above result reduces to a well-known extension of Farkas' Lemma for testing set inclusion of two polyhedra (Blanchini, 1999).

An immediate consequence of the above fact is that one can check whether (4.2) is true by solving a single LP (in fact, only the feasibility phase of an LP), where the number of constraints and decision variables scales polynomially with the size of the input data (the number of constraints and decision variables actually scales linearly with any single component of the data, if the size of all other components are fixed), without having to compute $\bigoplus_{i=0}^{k-1} L_i X$ explicitly.

Our next step to provide a more detailed computational procedure for a frequently encountered case in robust control of constrained linear discrete-time systems, namely the case when the disturbance set $\mathbb{W}$ is a polytope containing the origin and is given by

$$\mathbb{W} \triangleq \{w \in \mathbb{R}^n | Fw \leqslant g\}, \tag{4.4}$$

where $g \geqslant 0$ so that $\mathbb{W}$ contains the origin. Consider also the sets

$$\mathbb{X}(\mu, \beta, \delta) \triangleq \{x \in \mathbb{R}^n \mid H(x-\mu) \leqslant \beta r - \delta\}, \tag{4.5a}$$

$$\mathbb{U}(v, \gamma, \varepsilon) \triangleq \{u \in \mathbb{R}^m \mid P(u-v) \leqslant \gamma q - \varepsilon\}, \tag{4.5b}$$

where q, r, δ and ε are non-negative vectors, β and γ are non-negative scalars and H and P are matrices of compatible dimensions. The variables μ and v are such that the sets $\mathbb{X}(\mu, 1, 0) = \{x \mid H(x-\mu) \leqslant r\}$ and $\mathbb{U}(v, 1, 0) = \{u \mid P(u-v) \leqslant q\}$ with $r \geqslant 0$ and $q \geqslant 0$ correspond to the constraints $\mathbb{X}$ and $\mathbb{U}$ in (2.2); $\mu \in \mathbb{X}$ and $v \in \mathbb{U}$ are fixed for a given set of state and control constraints, β and γ are used to scale the sets around the pair (μ, v), δ and ε are used to decrease the right-hand side of the constraints. In order to ensure that $\mu \in \mathbb{X}(\mu, \beta, \delta)$ and $v \in \mathbb{U}(v, \gamma, \varepsilon)$, we require that $\beta r \geqslant \delta$ and $\gamma q \geqslant \varepsilon$.

For any $k \in \mathbb{N}_+$, recall that the matrices $\mathbf{M}_k \in \mathbb{R}^{km \times n}$ and $\mathscr{C}_i \in \mathbb{R}^{n \times km}$ are defined by (3.1) and (3.2), respectively. Following the discussion in Section 3.4, the variable $\sigma_k \triangleq (\alpha, \mathbf{M}_k, \bar{x}, \bar{u}, \beta, \gamma, \delta, \varepsilon)$ and the set Σ_k is defined as the set of σ_k that satisfies the following constraints:

$$\bar{x} = A\bar{x} + B\bar{u}, \quad \beta r \geqslant \delta, \quad \beta \in [0, 1], \quad \delta \geqslant 0, \tag{4.6a}$$

$$\gamma q \geqslant \varepsilon, \quad \gamma \in [0, 1], \quad \varepsilon \geqslant 0, \quad \alpha \in [0, 1), \tag{4.6b}$$

and the following subset inclusions

$$(A^k + \mathscr{C}_k \mathbf{M}_k)\mathbb{W} \subseteq \alpha \mathbb{W}, \tag{4.7a}$$

$$\bar{x} \oplus (1-\alpha)^{-1} \bigoplus_{i=0}^{k-1} (A^i + \mathscr{C}_i \mathbf{M}_k)\mathbb{W} \subseteq \mathbb{X}(\mu, \beta, \delta), \tag{4.7b}$$

$$\bar{u} \oplus (1-\alpha)^{-1} \bigoplus_{i=0}^{k-1} M_i \mathbb{W} \subseteq \mathbb{U}(v, \gamma, \varepsilon), \tag{4.7c}$$

where we recall that (4.7a) is (3.4) and (4.7b)–(4.7c) is equivalent to (3.19) if $\beta = 1$, $\gamma = 1$, $\delta = 0$ and $\varepsilon = 0$. Note that (4.7b) and (4.7c) are equivalent to

$$\bigoplus_{i=0}^{k-1} (A^i + \mathscr{C}_i \mathbf{M}_k)\mathbb{W} \subseteq (1-\alpha)\mathbb{X}(\mu - \bar{x}, \beta, \delta), \tag{4.8a}$$

$$\bigoplus_{i=0}^{k-1} M_i \mathbb{W} \subseteq (1-\alpha)\mathbb{U}(v - \bar{u}, \gamma, \varepsilon). \tag{4.8b}$$

It follows immediately from Proposition 4.1 that (4.7a) is equivalent to the existence of a matrix Λ such that

$$\Lambda g \leqslant \alpha g, \quad \Lambda F = F(A^k + \mathscr{C}_k \mathbf{M}_k), \quad \Lambda \geqslant 0, \tag{4.9}$$

where the vector and matrix inequalities are component-wise. By straightforward application of Proposition 4.1, it follows that (4.8a) is equivalent to the existence of a set of matrices $\Gamma \triangleq \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{k-1}\}$ such that

$$\sum_{i=0}^{k-1} \Gamma_i g \leqslant (1-\alpha)(\beta r - \delta + H\mu - H\bar{x}), \tag{4.10a}$$

$$\Gamma_i F = H(A^i + \mathscr{C}_i \mathbf{M}_k), \quad \Gamma_i \geqslant 0 \ \ \forall i \in \mathbb{N}_{k-1} \tag{4.10b}$$

and (4.8b) is equivalent to the existence of a set of matrices $\Pi \triangleq \{\Pi_0, \Pi_1, \ldots, \Pi_{k-1}\}$ such that

$$\sum_{i=0}^{k-1} \Pi_i g \leqslant (1-\alpha)(\gamma q - \varepsilon + Pv - P\bar{u}), \tag{4.11a}$$

$$\Pi_i F = P M_i, \quad \Pi_i \geqslant 0 \ \ \forall i \in \mathbb{N}_{k-1}. \tag{4.11b}$$

A direct consequence of the discussion above is:

Proposition 4.2. *Assume $k \in \mathbb{N}_+$ is chosen such that Σ_k is non-empty. Then $\sigma_k \in \Sigma_k$ (where $\sigma_k = (\alpha, \mathbf{M}_k, \bar{x}, \bar{u}, \beta, \gamma, \delta, \varepsilon)$) if and only if there exist a matrix Λ and sets of matrices $\Gamma \triangleq \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{k-1}\}$ and $\Pi \triangleq \{\Pi_0, \Pi_1, \ldots, \Pi_{k-1}\}$ such that constraints* (4.6), (4.9)–(4.11) *are satisfied.*

Constraints (4.10a) and (4.11a) are bilinear with respect to some of the components of σ_k, but all the other constraints are affine in all the components of σ_k, Λ, Γ and Π. However, for a fixed α, all the constraints are affine with respect to the remaining components in σ_k and all the components of Λ, Γ and Π. Hence, if α and k are fixed, then a $\sigma_k \in \Sigma_k$ can be computed by solving an LP. By inspection, it follows that the number of decision variables and constraints scales polynomially with the size of the input data, such as the number of states, inputs, constraints, etc. (the number of decision variables and constraints actually scales linearly with any single component of the data, if the size of all other components are fixed). Note that the number of constraints and decision variables also scale linearly with k, if the size of the other components are fixed.

4.2. The parameters $k, \alpha, \beta, \gamma, \delta, \varepsilon$ and suitable objective functions $f(\cdot)$

A suitable choice for a parameter $k \in \mathbb{N}$, as evident from Theorems 3.1–3.3, in the unconstrained as well as constrained case, is any integer $k \geqslant \mathscr{I}(A, B)$. By Theorems 3.1–3.2, there exists a collection of sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ that are RCI sets in the unconstrained case for any integer $k \geqslant \mathscr{I}(A, B)$. For the constrained case, it is necessary to determine a $k \in \mathscr{K}$, where $\mathscr{K}$ is defined in (3.21); this condition can be verified by solving a single optimization problem for any fixed $k \in \mathbb{N}$. However, we remark that for any integer $k \geqslant \mathscr{I}(A, B)$, in the constrained case one can obtain an upper bound on the magnitude of the disturbance such that there exists a collection of sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ that are RCI for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \lambda\mathbb{W})$, where λ can be used as a measure of the size of the allowable disturbance.

The parameter $\alpha \in [0, 1)$ is a contraction factor that allows for finite time parametrization of the RCI sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$

and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$. Its value can, in principle, be specified *a priori* by the designer, but it is possible to optimize it in certain cases.

The parameters $(\beta, \gamma) \in [0, 1] \times [0, 1]$ represent a relative contraction of the state and control constraint sets and are mainly included for the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$. In many control schemes, one would like to minimize the distance between possible trajectories of the uncertain system and the desired trajectory of a nominal system, consequently it is reasonable to attempt to construct a small (in an appropriately defined sense with respect to set inclusion) RCI set so that more freedom is allowed for generating a suitable nominal trajectory (Mayne et al., 2005; Raković, 2005). It is also possible to specify *a priori* values of the pair (β, γ); suitable values are $(\beta, \gamma) = (1, 1)$. The sets $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ are useful for the more complicated case when the origin is not necessarily in the interior of the state and control constraint sets. The relative contractions $(\beta, \gamma) \in [0, 1] \times [0, 1]$ of the state and control constraint sets for the family of $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ sets are replaced by relaxation variables δ and ε in order to enable a suitable pair $(\bar{x}, \bar{u}) \in \mathscr{Z}$ to be obtained through optimization. The values of the pair (δ, ε) can also be specified *a priori*, if desired; suitable values are $(\delta, \varepsilon) = (\mathbf{0}, \mathbf{0})$.

It is possible to specify a variety of objective functions $f(\cdot)$ by minor modification of the definition of the set Σ_k and to obtain tractable optimization problems (see, e.g., Raković, 2005; Raković & Mayne, 2005; Raković, Mayne, et al., 2005). For the case of the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$, we considered minimization of a specified polytopic norm of these sets (Raković, Mayne, et al., 2005) (an idea easily extendable to the sets $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$). For the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$, the objective function $f(\cdot)$ can be defined to be

$$f(\sigma_k) \triangleq q_\beta \beta + q_\gamma \gamma, \tag{4.12}$$

where the weights q_α and q_β express a preference for relative contraction of the state and control constraint sets. Similarly, for the case of the sets $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ it is perhaps desirable to minimize the Euclidean or any generalized distance between a desired operating point $(\hat{x}, \hat{u})$ and optimized equilibrium pair $(\bar{x}, \bar{u}) \in \mathscr{Z}$, so that one possible choice for the objective function $f(\cdot)$ is

$$f(\sigma_k) \triangleq |\bar{x} - \hat{x}|^2_{Q_\sigma} + |\bar{u} - \hat{u}|^2_{R_\sigma}, \tag{4.13}$$

where the positive definite weighting matrices Q_σ and R_σ are design variables. We remark that any (strictly) convex function in the decision variable σ_k (θ_k) is a suitable choice for the objective function. For any (strictly) convex function and for a fixed pair $(k, \alpha) \in \mathbb{N} \times [0, 1)$, the optimization problem $\mathbb{P}_k$ is a (*strict*) *convex programming problem*, and from Section 4.1 it follows that $\mathbb{P}_k$ is a standard *linear or quadratic programming problem* if the state, control and disturbance constraint sets are polytopic or polyhedral and the objective function is linear or quadratic (strictly convex), see for example (4.12) and (4.13). The objective function $f(\cdot)$ can be modified in order to extract a unique solution (Raković, 2005; Raković & Mayne, 2005; Raković, Mayne, et al., 2005), an appropriate cost function is a positively weighted quadratic norm of the decision variables σ_k (θ_k).

4.3. Implicit representation of the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$

Explicit computation of the Minkowski sum of sets is often numerically expensive and the complexity of the resulting output set, measured in terms of the number of facets and/or vertices, is not a polynomial function of the size of the input data, in general. However, in the case when the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ is known to be an RCI set for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ (e.g., as discussed before, by solving a suitably defined convex programming problem), then the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ can be represented implicitly without having to explicitly compute the Minkowski set additions involved in its definition. An alternative symbolic representation of the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ is

$$S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) = \{x \mid \exists \mathbf{w} \text{ s.t. } \mathbf{w} \in (1-\alpha)^{-1}\mathbf{W}^k, \bar{x} + \mathscr{D}_k \mathbf{w} = x\}.$$

The above observation allows one to represent the set $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ implicitly in x–$\mathbf{w}$ space, rather than computing it explicitly in x-space—this can be done by using standard computational geometry softwares (Kvasnica, Grieder, Baotić, & Morari, 2003; Veres, 2003). As before, the complexity of this implicit representation, measured in terms of the number of constraints and slack variables $\mathbf{w}$, also scales polynomially with the size of the input data, and linearly if the size of any single component is varied, but the size of the other components are fixed. Hence, checking whether $x \in S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ can be verified by solving a single, tractable LP if $\mathbb{W}$ is a polytope.

5. Illustrative examples

A theoretical comparison of the proposed procedure with the previous research that used $u = Kx$ is given in Raković (2005). In this case, the advantages of our method lie in the facts that: (i) the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ are RCI by construction for the unconstrained case (ii) hard state and control constraints are incorporated directly into the optimization problem and, (iii) the corresponding feedback control laws $\nu : R_{(k,\alpha)}(\mathbf{M}_k, \alpha) \to \mathbb{U}$ and $\mu : S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha) \to \mathbb{U}$ are set-valued and admit, in the general case, nonlinear selections. These advantages are also illustrated by a numerical example in Raković, Mayne, et al. (2005) and Raković (2005). Here we provide two additional illustrative examples.

The first example is an adequate variation of the example considered in Blanchini (1994), with a change that the parametric and additive uncertainty appearing in the system description in Blanchini (1994) are converted into a single additive uncertainty:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w,$$

where the hard state, control and disturbance constraint sets are

$$\mathbb{X} = \mathbb{B}^2_\infty(1), \quad \mathbb{U} = \{u \mid |u| \leqslant 1\}, \quad \mathbb{W} = \begin{bmatrix} \frac{2}{10} & \frac{1}{10} \\ 0 & \frac{1}{10} \end{bmatrix} \mathbb{B}^2_\infty(1).$$

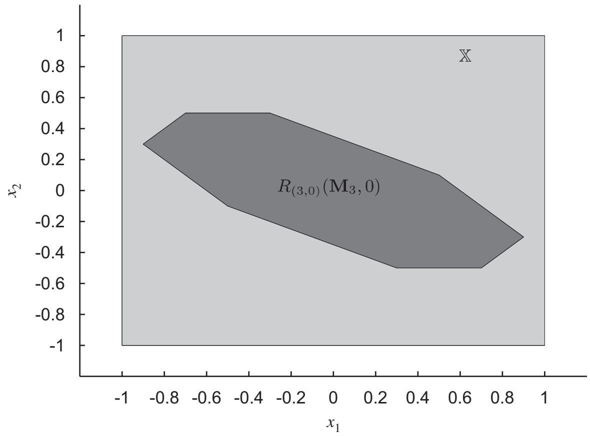

Fig. 1. The RCI set $R_{(3,0)}(\mathbf{M}_3, 0)$.

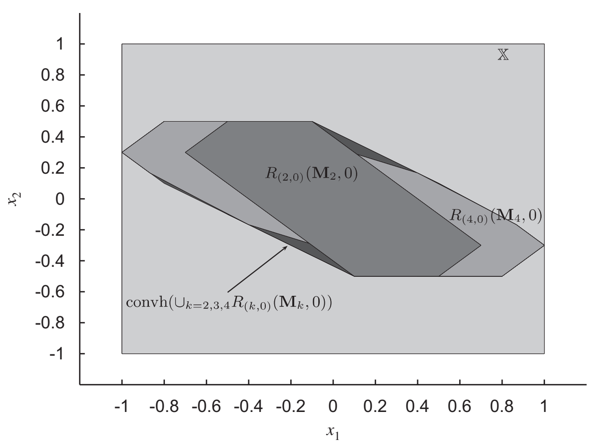

Fig. 2. The RCI sets $R_{(2,0)}(\mathbf{M}_2, 0)$, $R_{(4,0)}(\mathbf{M}_4, 0)$ and $\mathrm{convh}(\bigcup_{k=2,3,4} R_{(k,0)}(\mathbf{M}_k, 0))$.

For this example, the RCI sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ exist for all $k \in \mathbb{N}$, $k \geqslant 2$. Fig. 1 shows an RCI set $R_{(3,0)}(\mathbf{M}_3, 0)$ obtained by an LP, as outlined in Section 4.1, for $k=3$ and with $(\bar{x}, \bar{u}, \alpha, \delta, \varepsilon) = (\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0})$, $(\mu, v) = (\mathbf{0}, \mathbf{0})$ and by using objective function (4.12) with $(q_\beta, q_\gamma) = (0, 1)$. The solution of the LP yielded a matrix $\mathbf{M}_3$ given by

$$\mathbf{M}_3 = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix}^{\mathrm{T}},$$

and the RCI set $R_{(3,0)}(\mathbf{M}_3, 0)$ obtained by (3.3). Fig. 2 shows RCI sets $R_{(2,0)}(\mathbf{M}_2, 0)$ and $R_{(4,0)}(\mathbf{M}_4, 0)$, obtained by solving the corresponding LPs with the same ingredients as above for $k = 2$ and 4, as well as the RCI set $\mathrm{convh}(\bigcup_{k=2,3,4} R_{(k,0)}(\mathbf{M}_k, 0))$. The robust control invariant sets $\mathrm{convh}(\bigcup_{k=2,3,4} R_{(k,0)}(\mathbf{M}_k, 0))$, $R_{(2,0)}(\mathbf{M}_2, 0)$, $R_{(3,0)}(\mathbf{M}_3, 0)$ and $R_{(4,0)}(\mathbf{M}_4, 0)$ have, respectively, 11, 6, 8 and 14 vertices.

Our second example illustrates the relationship between the RCI sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ and the maximal RCI set for the system (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$. Consider a very simple two-dimensional system:

$$x^+ = x + u + w, \tag{5.1}$$

with $\mathbb{X} = \mathbb{B}_1^2(1) \oplus \mathbb{B}_\infty^2(1)$, $\mathbb{U} = \mathbb{B}_1^2(2)$, $\mathbb{W} = \mathbb{B}_\infty^2(1)$. The matrices M_0 and M_1 given by

$$M_0 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

are such that $M_0 \mathbb{B}_\infty^2(1) \oplus M_1 \mathbb{B}_\infty^2(1) = \mathbb{B}_1^2(2)$, $(A + BM_0)\mathbb{B}_\infty^2(1) = \mathbb{B}_1^2(1)$ and $(A^2 + ABM_0 + BM_1)\mathbb{B}_\infty^2(1) = \{0\}$ (with $A = I$ and $B = I$) so that the set $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ defined for $k = 2$, $\alpha = 0$ and $\mathbf{M}_k$ obtained from matrices M_0 and M_1 and that takes the particular form $(A + BM_0)\mathbb{B}_\infty^2(1) \oplus \mathbb{B}_\infty^2(1)$ is equal to $\mathbb{X}$—the maximal RCI set in this case.

However, the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ cannot in general capture the geometry of the maximal RCI set as illustrated by considering the same system (5.1) with $\mathbb{X} = \mathbb{B}_1^2(2)$, $\mathbb{U} = \mathbb{B}_1^2(2)$, $\mathbb{W} = \mathbb{B}_\infty^2(1)$. By direct inspection, the maximal RCI set is $\mathbb{X}$. The RCI sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$, $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$, $\mathscr{S}_{\mathscr{K}}$ and $\mathscr{S}_{\mathscr{K}}^C$ are, in this example, all identically equal to a single set (regardless of the choice of $k \in \mathbb{N}$) that is equal to $\mathbb{W}$ and is a RCI set for this case. A limitation illustrated by this example is a consequence of the impossibility to decompose the set $\mathbb{X} = \mathbb{B}_1^2(2)$ with respect to Minkowski set addition whose summands are $\mathbb{W} \triangleq \mathbb{B}_\infty^2(1)$ and its linear transformations. A simple way to overcome this limitation (in view of Remark 1) is to consider the artificial disturbance set $\tilde{\mathbb{W}} \triangleq \mathbb{X}$ that together with $M_0 = -I$, leads to the RCI set $\tilde{R}_{(1,0)}(M_0, 0)$ equal to $\mathbb{X}$. Nevertheless, since the sets $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$ are described by the Minkowski sums (modulo translation involved in the definition of the sets $S_{(k,\alpha)}(\bar{x}, \mathbf{M}_k, \alpha)$) with summands of the form $(A^i + \mathscr{C}_i \mathbf{M}_k)\mathbb{W}$, $i \in \mathbb{N}_{k-1}$ there is, most probably, no guarantee that these sets can capture the geometry of the maximal RCI set.

6. Concluding remarks

In this paper we established the existence of two families of robust control invariant sets for which the corresponding control law is nonlinear (piecewise affine in the most frequently encountered cases) enabling better results to be obtained compared with existing methods where the control law is linear (Kolmanovsky & Gilbert, 1998; Raković, Kerrigan, et al., 2005). Construction of a member of these families satisfying an appropriate criterion can be obtained from the solution of an appropriately specified LP when the additive disturbance, state and control constraints are polytopic. Another advantage over some of the existing methods is the fact that the proposed algorithms do not require standard iterative set computations. Under some mild assumptions, it is sufficient to solve a single *convex programming problem*. The *optimized robust control invariance* algorithms were illustrated by some adequate examples.

Future research directions are to investigate more thoroughly the relationship between the parametrized robust control invariant sets and the maximal robust control invariant set. The results presented in this paper can be extended to the case when the system dynamics are subject to bounded parametric uncertainty. It is, in principle, possible to extend the results of this paper by exploiting direct enumeration usually employed in the case of parametrically uncertain systems. However, this extension needs careful investigation since the structure of uncertainty is different and might require modification of the presented methods. Finally, an extension of the proposed approach to the case when the controllability assumption is replaced by a weaker stabilizability assumption is possible by combining the presented results and the results established in Raković, Kerrigan, et al. (2005).

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Appendix A

A.1. Proof of Theorem 3.1

The fact that $\mathbb{M}_{(k,\alpha)} \neq \emptyset$ for all $k \in \mathbb{N}$, $k \geqslant \mathscr{I}(A, B)$ is a direct consequence of Assumption 2.1. The second claim is established as follows. Fix $k \in \mathbb{N}$ such that $\mathbb{M}_{(k,\alpha)} \neq \emptyset$ and let $(\mathbf{M}_k, \alpha) \in \mathbb{M}_{(k,\alpha)}$. Let x be an arbitrary element of $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$. Since $x \in R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ it follows by the definition of the set $R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$:

$$\begin{aligned} x &= (A^{k-1} + A^{k-2}BM_0 + \cdots + BM_{k-2})w_0 \\ &\quad + (A^{k-2} + A^{k-3}BM_0 + \cdots + BM_{k-3})w_1 \\ &\quad + \cdots + (A + BM_0)w_{k-2} + w_{k-1} \end{aligned} \tag{A.1}$$

for some $w_i \in (1-\alpha)^{-1}\mathbb{W}, i \in \mathbb{N}_{k-1}$. For each $x \in R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ let $\mathbf{W}_\alpha(x)$ be defined by

$$\mathbf{W}_\alpha(x) \triangleq \{\mathbf{w} \mid \mathbf{w} \in (1-\alpha)^{-1}\mathbf{W}^k,\ \mathscr{D}_k\mathbf{w} = x\}, \tag{A.2}$$

where $\mathbf{w} \triangleq \{w_0, w_1, \ldots, w_{k-1}\}$ (with vectorized form $\mathbf{w} = [w_0^{\mathrm{T}}\ w_1^{\mathrm{T}}\ \cdots\ w_{k-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{kn}$ used in algebraic expressions), $\mathbf{W}^k \triangleq \mathbb{W} \times \mathbb{W} \times \cdots \times \mathbb{W}$ and $\mathscr{D}_k$ is given by (3.13). The set-valued map $\mathbf{W}_\alpha(x)$ is non-empty for all $x \in R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ by the definition of the Minkowski set addition. Let $\mathbf{w}(x)$ be a point-valued selection satisfying $\mathbf{w}(x) \in \mathbf{W}_\alpha(x)$ (an appropriate selection can be defined as in (3.17)). Let the feedback control law $v : R_{(k,\alpha)}(\mathbf{M}_k, \alpha) \to \mathbb{R}^m$ be defined by

$$v(x) = M_{k-1}w_0(x) + M_{k-2}w_1(x) + \cdots + M_0 w_{k-1}(x). \tag{A.3}$$

From (A.1)–(A.3), for any arbitrary $x \in R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ and any arbitrary $w \in \mathbb{W}$ we have

$$\begin{aligned} x^+ &= Ax + Bv(x) + w \\ &= (A^k + A^{k-1}BM_0 + \cdots + BM_{k-1})w_0(x) \\ &\quad + (A^{k-1} + A^{k-2}BM_0 + \cdots + BM_{k-2})w_1(x) \\ &\quad + \cdots + (A + BM_0)w_{k-1}(x) + w. \end{aligned} \tag{A.4}$$

Therefore,

$$x^+ \in (1-\alpha)^{-1} \bigoplus_{i=1}^{k} (A^i + \mathscr{C}_i\mathbf{M}_k)\mathbb{W} \oplus \mathbb{W}. \tag{A.5}$$

From Assumption 2.1 and since $(\mathbf{M}_k, \alpha) \in \mathbb{M}_{(k,\alpha)}$ it follows (see (3.4)) that $(1-\alpha)^{-1}(A^k + \mathscr{C}_k\mathbf{M}_k)\mathbb{W} \subseteq \alpha(1-\alpha)^{-1}\mathbb{W}$ and $(1-\alpha)^{-1}(A^k + \mathscr{C}_k\mathbf{M}_k)\mathbb{W} \oplus \mathbb{W} \subseteq (1-\alpha)^{-1}\mathbb{W}$, which when combined with (A.5) yields

$$x^+ \in (1-\alpha)^{-1} \bigoplus_{i=0}^{k-1} (A^i + \mathscr{C}_i\mathbf{M}_k)\mathbb{W}. \tag{A.6}$$

Hence $x^+ \in Ax + Bv(x) \oplus \mathbb{W} \subseteq R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ for all $x \in R_{(k,\alpha)}(\mathbf{M}_k, \alpha)$ with $v(x)$ defined by (A.2) and (A.3).

A.2. Proof of Proposition 3.1

Let $x \in \Omega_{\bar{x}} \triangleq \bar{x} \oplus \Omega$, $\bar{x} \in \mathscr{X}$ so that $x = \bar{x} + y$ for some $y \in \Omega$. We have $x^+ \in Ax + Bu \oplus \mathbb{W} = A(\bar{x} + y) + Bu \oplus \mathbb{W}$. Let $u = \bar{u} + v$, where $\bar{u}$ is any $\bar{u} \in \mathbb{U}(\bar{x})$ with $\mathbb{U}(\bar{x})$ given by (3.9). It follows that $x^+ \in A\bar{x} + B\bar{u} + Ay + Bv \oplus \mathbb{W}$. Since $\bar{u} \in \mathbb{U}(\bar{x})$ we have $\bar{x} = A\bar{x} + B\bar{u}$ (because$(\bar{x}, \bar{u}) \in \mathscr{Z}$) and $x^+ \in \bar{x} + Ay + Bv \oplus \mathbb{W}$. Since $y \in \Omega$ and Ω is a RCI set for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, \mathbb{W})$ it follows that for any $y \in \Omega$ there exists a $v \in \mathbb{R}^m$ such that $Ay + Bv \oplus \mathbb{W} \subseteq \Omega$ and consequently $x^+ \in \bar{x} + Ay + Bv \oplus \mathbb{W} \subseteq \bar{x} \oplus \Omega = \Omega_{\bar{x}}$ for all $(x, w) \in \Omega_{\bar{x}} \times \mathbb{W}$.

A.3. Proof of Proposition 3.2

Convexity of the set Θ_k for a suitable, fixed, $(k, \alpha) \in \mathbb{N} \times [0, 1)$ is established as follows. Suppose that given the pair $(k, \alpha) \in \mathbb{N} \times [0, 1)$ such that the set Θ_k is non-empty and is not a singleton (since if it is a singleton the claim is trivially true) and let $\theta_k^l = (\bar{x}^l, \bar{u}^l, \mathbf{M}_k^l, \alpha)$, $l = 1, 2$, be any two distinct points such that $\theta_k^l \in \Theta_k$, $l = 1, 2$. Since $\theta_k^l \in \Theta_k$, $l = 1, 2$ we have

$$\bar{x}^l \oplus (1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} (A^j + \mathscr{C}_j\mathbf{M}_k^l)\mathbb{W} \subseteq \mathbb{X},$$

$$\bar{u}^l \oplus (1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} M_j^l\mathbb{W} \subseteq \mathbb{U},$$

$$\bar{x}^l = A\bar{x}^l + B\bar{u}^l \quad \text{and} \quad (A^k + \mathscr{C}_k\mathbf{M}_k^l)\mathbb{W} \subseteq \alpha\mathbb{W} \tag{A.7}$$

for $l=1,2$. Let $\lambda \triangleq (\lambda_1, \lambda_2)$ and $\Lambda \triangleq \{\lambda \in \mathbb{R}^2 \mid \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}$. Let also θ_k^λ be defined, for any arbitrary $\lambda \in \Lambda$, by

$$\theta_k^\lambda \triangleq \lambda_1 \theta_k^1 + \lambda_2 \theta_k^2, \tag{A.8}$$

so that $\theta_k^\lambda = (\bar{x}^\lambda, \bar{u}^\lambda, \mathbf{M}_k^\lambda, \alpha^\lambda)$, where

$$\bar{x}^\lambda \triangleq \lambda_1 \bar{x}^1 + \lambda_2 \bar{x}^2, \quad \bar{u}^\lambda \triangleq \lambda_1 \bar{u}^1 + \lambda_2 \bar{u}^2,$$

$$\mathbf{M}_k^\lambda \triangleq \lambda_1 \mathbf{M}_k^1 + \lambda_2 \mathbf{M}_k^2, \quad \alpha^\lambda \triangleq \lambda_1 \alpha + \lambda_2 \alpha = \alpha. \tag{A.9}$$

We show that $\theta_k^\lambda \in \Theta_k$ for any arbitrary $\lambda \in \Lambda$. We recall that for any matrix pair $(F, G) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ and any convex set $\mathscr{A} \subseteq \mathbb{R}^n$ the set inclusion

$$(F+G)\mathscr{A} \subseteq F\mathscr{A} \oplus G\mathscr{A} \tag{A.10}$$

is one of the basic rules of Minkowski algebra of convex bodies (see, e.g., Schneider, 1993). We also have that, for all $j \in \mathbb{N}_{k-1}$,

$$A^j + \mathscr{C}_j \mathbf{M}_k^\lambda = \lambda_1 (A^j + \mathscr{C}_j \mathbf{M}_k^1) + \lambda_2 (A^j + \mathscr{C}_j \mathbf{M}_k^2). \tag{A.11}$$

Combining (A.10) and (A.11) we have

$$\begin{aligned}&(A^j + \mathscr{C}_j \mathbf{M}_k^\lambda)\mathbb{W}\\ &\quad \subseteq \lambda_1 (A^j + \mathscr{C}_j \mathbf{M}_k^1)\mathbb{W} \oplus \lambda_2 (A^j + \mathscr{C}_j \mathbf{M}_k^2)\mathbb{W}\end{aligned} \tag{A.12}$$

for all $j \in \mathbb{N}_{k-1}$ so that

$$\begin{aligned}&(1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} (A^j + \mathscr{C}_j \mathbf{M}_k^\lambda)\mathbb{W}\\ &\quad \subseteq \lambda_1 (1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} (A^j + \mathscr{C}_j \mathbf{M}_k^1)\mathbb{W}\\ &\qquad \oplus \lambda_2 (1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} (A^j + \mathscr{C}_j \mathbf{M}_k^2)\mathbb{W}.\end{aligned} \tag{A.13}$$

From (A.7)–(A.9) and (A.13) we have

$$\bar{x}^\lambda \oplus (1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} (A^j + \mathscr{C}_j \mathbf{M}_k^\lambda)\mathbb{W} \subseteq \mathbb{X}. \tag{A.14}$$

A similar argument (A.10)–(A.14) yields

$$\begin{aligned}&\bar{u}^\lambda \oplus (1-\alpha)^{-1} \bigoplus_{j=0}^{k-1} M_j^\lambda \mathbb{W} \subseteq \mathbb{U} \quad \text{and}\\ &(A^k + \mathscr{C}_k \mathbf{M}_k^\lambda)\mathbb{W} \subseteq \alpha \mathbb{W}.\end{aligned} \tag{A.15}$$

Since $\bar{x}^\lambda = A\bar{x}^\lambda + B\bar{u}^\lambda$ is clearly true, it follows that θ_k^λ (defined by (A.8)–(A.9)) satisfies $\theta_k^\lambda \in \Theta_k$ for any arbitrary $\lambda \in \Lambda$.

The second property of the set Θ_k is easily established by induction. Suppose that $k \in \mathbb{N}$ is such that Θ_k is non-empty with $\alpha = 0$ fixed. Let $\theta_k = (\bar{x}, \bar{u}, \mathbf{M}_k, 0)$ be any $\theta_k \in \Theta_k$ (with $\alpha = 0$ fixed). To establish that the set Θ_{k+1} is non-empty we verify that $\bar{\theta}_{k+1} \triangleq (\bar{x}, \bar{u}, \mathbf{M}_{k+1}, 0)$ with $\mathbf{M}_{k+1} \triangleq (\mathbf{M}_k, \mathbf{0})$, where $\mathbf{0}$ is a zero matrix of appropriate dimensions, satisfy $\bar{\theta}_{k+1} \in \Theta_{k+1}$. Since $A^k + \mathscr{C}_k \mathbf{M}_k = \mathbf{0}$ we have

$$A^{k+1} + \mathscr{C}_{k+1}\mathbf{M}_{k+1} = A(A^k + \mathscr{C}_k \mathbf{M}_k) + B\mathbf{0} = \mathbf{0}. \tag{A.16}$$

Hence $(A^{k+1} + \mathscr{C}_{k+1}\mathbf{M}_{k+1})\mathbb{W} = \{0\}$. A direct verification yields

$$S_{(k+1,0)}(\bar{x}, \mathbf{M}_{k+1}, 0) \subseteq \mathbb{X} \tag{A.17}$$

and

$$U(\bar{u}, \mathbf{M}_{k+1}, 0) \subseteq \mathbb{U}. \tag{A.18}$$

It follows that $\bar{\theta}_{k+1} \triangleq (\bar{x}, \bar{u}, \mathbf{M}_{k+1}, 0)$ with $\mathbf{M}_{k+1} \triangleq (\mathbf{M}_k, \mathbf{0})$ satisfies $\bar{\theta}_{k+1} \in \Theta_{k+1}$. The remaining assertions follow by induction.

References

Aubin, J. P. (1991). Viability theory. *Systems & control: Foundations & applications*. Boston, Basel, Berlin: Birkhauser.

Bemporad, A., & Morari, M. (1999). Robust model predictive control: A survey. In *Robustness in identification and control. Lecture notes in control and information sciences* (Vol. 245, pp. 207–226). Berlin: Springer (survey paper).

Bemporad, A., Morari, M., Dua, V., & Pistikopoulos, E. (2002). The explicit linear quadratic regulator for constrained systems. *Automatica*, *38*, 3–20.

Bertsekas, D. P. (1972). Infinite-time reachability of state-space regions by using feedback control. *IEEE Transactions on Automatic Control*, *17*(5), 604–613.

Bertsekas, D. P., & Rhodes, I. B. (1971). On the minimax reachability of target sets and target tubes. *Automatica*, *7*, 233–247.

Blanchini, F. (1994). Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Transactions on Automatic Control*, *39*(2), 428–433.

Blanchini, F. (1999). Set invariance in control. *Automatica*, *35*, 1747–1767 (survey paper).

Blanchini, F., Mesquine, F., & Miani, S. (1995). Constrained stabilization with an assigned initial condition set. *International Journal of Control*, *62*(3), 601–617.

Caravani, P., & De Santis, E. (2000). A polytopic game. *Automatica*, *36*, 973–981.

Caravani, P., & De Santis, E. (2002). Doubly invariant equilibria of linear discrete-time games. *Automatica*, *38*, 1531–1538.

da Silva, J. G., Jr., & Tarbouriech, S. (1999). Invariance and contractivity of polyhedra for linear continuous-time systems with saturating controls. *Revista Controle e Automação da Sociedade Brasileira de Automática (SBA)*, *10*(3), 149–158.

De Santis, E., Di Benedetto, M. D., & Berardi, L. (2004). Computation of maximal safe sets for switching systems. *IEEE Transactions on Automatic Control*, *49*(2), 184–195.

Dórea, C. E. T., & Hennet, J. C. (1999). (A,B)—Invariant polyhedral sets of linear discrete-time systems. *Journal of Optimization Theory and Applications*, *103*(3), 521–542.

Findeisen, R., Imsland, L., Allgöwer, F., & Foss, B. A. (2003). State and output feedback nonlinear model predictive control: An overview. *European Journal of Control*, *9*(2–3), 190–206 (survey paper).

Gilbert, E. G., & Kolmanovsky, I. (1999). Fast reference governors for systems with state and control constraints and disturbance inputs. *International Journal of Robust and Nonlinear Control*, *9*(15), 1117–1141.

Kolmanovsky, I., & Gilbert, E. G. (1998). Theory and computation of disturbance invariance sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, *4*, 317–367.

Kvasnica, M., Grieder, P., Baotić, M., & Morari, M. (2003). Multi parametric toolbox (MPT). In *Hybrid systems: Computation and control. Lecture notes in computer science*. Berlin: Springer.

La Salle, J. P. (1976). The stability of dynamical systems. *CBMS-NSF regional conference series in applied mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, with Appendix "Limiting Equations and Stability of Nonautonomous Ordinary Differential Equations" by Z. Artstein.

Mayne, D. Q. (2001). Control of constrained dynamic systems. *European Journal of Control*, *7*, 87–99 (survey paper).

Mayne, D. Q., & Schroeder, W. R. (1997). Robust time-optimal control of constrained linear systems. *Automatica*, *33*, 2103–2118.

Mayne, D. Q., Seron, M., & Raković, S. V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, *41*, 219–224.

Raković, S. V. (2005). *Robust control of constrained discrete time systems: Characterization and implementation*. Ph.D. thesis, Imperial College London, London, United Kingdom.

Raković, S. V., De Santis, E., & Caravani, P. (2005). Invariant equilibria of polytopic games via optimized robust control invariance. In *Proceedings of the 44th IEEE conference on decision and control*. Seville, Spain.

Raković, S. V., Kerrigan, E. C., Kouramas, K. I., & Mayne, D. Q. (2005). Invariant approximations of the minimal robustly positively invariant set. *IEEE Transactions on Automatic Control*, *50*(3), 406–410.

Raković, S. V., & Mayne, D. Q. (2005). Regulation of discrete-time linear systems with positive state and control constraints and bounded disturbances. In *Proceedings of the 16th IFAC world congress IFAC 2005*, Praha, Czech Republic.

Raković, S. V., Mayne, D. Q., Kerrigan, E. C., & Kouramas, K. I. (2005). Optimized robust control invariant sets for constrained linear discrete-time systems. In *Proceedings of the 16th IFAC world congress IFAC 2005*. Praha, Czech Republic.

Rockafellar, R. T. (1970). *Convex analysis*. Princeton, NJ: Princeton University Press.

Schneider, R. (1993). *Convex bodies*: *The Brunn–Minkowski theory. Encyclopedia of mathematics and its applications* (Vol. 44). Cambridge: Cambridge University Press.

Veres, S. M. (2003). Geometric Bounding Toolbox (GBT) for MATLAB. Official website: ⟨http://www.sysbrain.com⟩.

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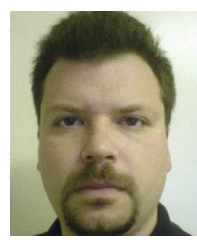

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