Conic programming approach for static voltage stability analysis in radial networks

R.A. Jabr and B.C. Pal

Abstract: That the problem of computing the capacity limit of a radial distribution system can be formulated as a second-order cone program is shown. The implications of the conic programming formulation are 2-fold. First, the load capability of the radial system can be obtained using existing efficient implementations of polynomial time interior-point algorithms, thus avoiding the need for running a sequence of load flow solutions. Secondly, the conic objective function yields a voltage stability indicator (SI). This indicator quantifies the maximum percentage by which the current load profile can be uniformly increased before voltage collapse occurs. The proposed method is validated by computing the load capability and voltage SIs of 11 different distribution systems. Comparisons are carried out with five previously published voltage SIs.

1 Introduction

Power distribution networks are being constantly faced with an ever increasing load demand. Moreover, with the advent of deregulation in the power industry, there is a greater focus on managing network assets efficiently rather than reinforcing the network’s capacity. This results in the networks being operated closer to their transmission capacity. It is well known that heavily loaded distribution networks operate in the vicinity of their steady state power transfer limit, thus making them prone to voltage collapse. The problem of determining the load capability and the corresponding voltage stability margins is therefore of major concern to power distribution engineers.

The static voltage stability problem has been classically related to the reaching of some maximum admissible load, beyond which a load flow solution no longer exists [1]. In fact, early methods for determining the stability limit relied on solving a sequence of load flow solutions for successively increasing multiples of the base load vector. In Venikov et al. [2], the static voltage instability condition is said to be reached at the first multiple of the base load vector for which the load flow algorithm fails to converge. Divergence of the load flow algorithm, however, may not be necessarily attributed to the instability condition beyond the critical point. It may be caused by numerical ill-conditioning close to the voltage stability limit (critical point). To mitigate this problem, a continuation load flow which remains well conditioned at and around the critical point was proposed in Ajjarapu and Christy [3]. The predictor–corrector technique in Ajjarapu and Christy [3] was also recently extended for voltage stability analysis of unbalanced three-phase systems [4]. The load flow based methods require solving a large number of load flow problems and are therefore computationally expensive. In an attempt to produce more efficient techniques, several researchers proposed static voltage stability indicators (SIs) that yield an estimate of how far a given operating condition is from the stability limit. One such indicator is the minimum singular value of the power flow Jacobian matrix [5]. Canizares et al. [6] argue that the minimum singular value may not be a very good indicator of proximity to the collapse point because of its nonlinear behaviour. They [6] also discuss improved test functions. Other indicators require reduction of the system into a two-bus equivalent [7, 8]. The indicators in Ajjarpu and Christy [3], Löff et al. [5], Jasmon and Lee [7] and Moghavvemi and Faraque [8] all propose using a single index for the whole network. Such an index can be easily computed. It is possible, however, to compute a SI for every single bus or line in the network. For instance, in Chebbo et al. [1], a bus voltage SI based on the theorems of the Thévenin equivalent and maximum power transfer is proposed. The line voltage SIs include those based on the reactive power flow [9] and the bi-quadratic equation in radial networks [10].

The voltage SI proposed in this paper is based on the computation of the system load capability via an optimisation technique. This computation, also known as the transfer capability calculation, has been recently discussed in an optimisation framework that includes both security constraints and FACTS controllers [11]. The optimisation problem in Zhang [11] was solved via a nonlinear interior-point method. An earlier study considered system load capability computation of unbalanced radial networks [12]. The algorithm in Miu and Chiang [12] is based on a combination of a current-based estimator, a voltage-based estimator and a three-phase power flow.

The optimisation technique that is proposed in this study for the computation of the system load capability is known as conic programming. In fact, conic programming has been recently proposed to solve the radial load flow problem [13]. This paper extends the conic radial load flow method for the computation of the system load capability. The preliminary algorithm development and testing are limited to a balanced radial network with the loads being modelled as constant power. The main advantage of...
the proposed approach is that it can make use of existing high-powered software tools such as MOSEK [14] for solving conic programming problems.

2 Second-order cone programming

2.1 Definition of a cone program

One way of generalising a linear optimisation problem [15] is to include a constraint of the form

\[ x \in C \]

in the problem definition, where \( C \) is required to be a convex cone [16]. The resulting problem is called a conic program. Let each element \( x_i \) of the vector \( x \) be a member of exactly one of the vectors \( x^j, j = 1, \ldots, k \). Condition (1) is satisfied if each one of the vectors \( x^j \) belongs to one of the following cones.

1. the \( R \) set (set of real numbers)
2. the quadratic cone

\[ C^q = \left\{ x \in \mathbb{R}^m : x_1 \geq \sqrt{\sum_{j=2}^{m} x_j^2} \right\} \]

3. the rotated quadratic cone

\[ C_r = \left\{ x \in \mathbb{R}^m : 2x_1x_2 \geq \sum_{j=3}^{m} x_j^2, x_1, x_2 \geq 0 \right\} \]

Conic quadratic problems can be solved by polynomial time interior-point methods at basically the same computational complexity as linear programming problems of similar size. For details on the conic programming implementation, the interested reader can refer to Anderson et al. [16]. A high quality implementation of a primal-dual interior-point method for quadratic conic programming is available in the MOSEK [14] software package. MOSEK was used to solve the conic programs in this research.

2.2 Radial load-flow analysis

Consider a radial distribution network with one substation connected at node 1 and load nodes numbered as \( 2, \ldots, N \). It is assumed that the magnitude of the voltage at node 1 is specified. Denote by \( a(i) \) the set of nodes connected to node \( i \) and by \( P_{Li} / Q_{Li} \) the real/reactive power loads (in pu) at node \( i \). Let the line model consist of the single-line equivalent circuit shown in Fig. 1 (all relevant quantities are in pu) and define \( \theta_{ij} = \theta_i - \theta_j \), \( u_i = \frac{V_i^2}{\sqrt{2}}, R_i = V_i \cos \theta_i \) and \( I_i = V_i \sin \theta_i \).

It has been shown in Jabr [13] that the radial load-flow solution can be obtained by solving the following second-order cone program

\[ \text{maximize } \sum_{j \text{ lines}} R_{ij} u_j \text{ subject to} \]

1. For \( i = 2, \ldots, N \)

\[ -\sqrt{2}u_i \sum_{j \in a(i)} G_{ij} + \sum_{j \in a(i)} (G_{ij} R_{ij} - B_{ij} I_{ij}) - \lambda P_{Li} = 0 \]

(11)

\[ -\sqrt{2}u_i \sum_{j \in a(i)} B_{ij} + \sum_{j \in a(i)} (B_{ij} R_{ij} + G_{ij} I_{ij}) - \lambda Q_{Li} = 0 \]

(12)

\[ u_i \geq 0 \text{ and } u_1 = \frac{V_1^2}{\sqrt{2}} \]

(13)

2. For all \( ij \) lines: (8) and (9).

Note that if it were not for the rotated quadratic cone constraints in (8), the above program would have been a linear optimisation problem. The original variables (i.e. \( V_j \) and \( \theta_j \)) can be easily deduced once the new adopted variables (i.e. \( u_i, R_{ij} \) and \( I_{ij} \)) are computed.

2.3 Radial load capability analysis

The problem of computing the system load capability is formulated for an arbitrary load variation pattern where the (constant power factor) loads in the pattern change by the same percentage. This assumption on load variation has been adopted in Ajjarapu and Christy [3], Chakravorty and Das [10], Zhang [11] and Miu and Chiang [12]. Let \( \lambda \in R \) denote the load variation factor (\( \lambda = 1 \) corresponds to the base load). The conic radial load-flow formulation in Section 2.2 can be extended for computation of the load capability. The desired conic program is:

\[ \text{maximize } \lambda \text{ subject to} \]

1. For \( i = 2, \ldots, N \)

\[ -\sqrt{2}u_i \sum_{j \in a(i)} G_{ij} + \sum_{j \in a(i)} (G_{ij} R_{ij} - B_{ij} I_{ij}) - \lambda P_{Li} = 0 \]

(11)

\[ -\sqrt{2}u_i \sum_{j \in a(i)} B_{ij} + \sum_{j \in a(i)} (B_{ij} R_{ij} + G_{ij} I_{ij}) - \lambda Q_{Li} = 0 \]

(12)

\[ u_i \geq 0 \text{ and } u_1 = \frac{V_1^2}{\sqrt{2}} \]

(13)

2. For all \( ij \) lines: (8) and (9).

From (11) and (12), it is evident that increasing the objective in (10) calls for increasing the values of \( R_{ij} \). This results in (8) being active at optimality, a condition required for a valid load-flow solution [13]. Moreover, for networks that admit a load-flow solution for the base load profile, the optimal objective function in (10) would be greater than or equal to one.

It is worthy to note that the above formulation can also account for load voltage limits and real/reactive power line flow limits by simply adding the relevant constraints. For the purpose of performing a comparative study with the sequential load-flow (SLF) approach [10], these additional constraints were not included. Notwithstanding,
it should be clear that both the SLF approach and the conic program produce estimates of the system load capability and voltage margin. This is because as voltage falls, loads behave nonlinearly. Unfortunately, knowledge of the load behaviour below about 0.9 pu voltage is extremely sparse [17].

3 Static voltage SI

3.1 Loading factor SI

Let the optimal load variation factor obtained from solving the conic problem in Section 2.3 be denoted by $\lambda^*$. The margin ($\lambda^* - 1$) can be used as a SI. This indicator implies that the base load profile can be uniformly increased by $100 \times (\lambda^* - 1)\%$ to reach the voltage stability limit or the critical point.

Alternatively, it is possible to define an indicator

$$L_c = \frac{1}{\lambda}$$

(14)

For networks that admit a power flow solution for the base load profile, $L_c$ is always bounded between zero and one. A value of $L_c$ close to zero implies that for the given loading pattern, the network’s operating point is far from its stability limit. Whereas a value close to one indicates that the base load profile is beyond the system load capability.

3.2 Other SIs

For comparative purposes, other recently published SIs are detailed herein. These indicators are as follows.

1. The bus SI proposed by Chebbo et al. [1]. This indicator is obtained from calculating the magnitude of the ratio of the Thévenin to load impedances at a given bus-bar. The Thévenin impedance at the $j$th bus-bar is the $j$th diagonal element of the bus impedance matrix constructed after replacing all real/reactive loads by their equivalent impedances. When a single load change is considered, the critical stability limit is reached once the bus indicator becomes one. For system load changes, the SI can be defined as the maximum value of the bus SI computed at all bus-bars. $L_{mn}$ is the line SI proposed by Chakravorty and Das [10]. This indicator is based on Csendes’ radial load-flow [18], which defines a bi-quadratic equation relating the voltage magnitudes at the sending and receiving ends of each branch to the power flow at the receiving end. The critical point in this case is reached when the discriminant of the bi-quadratic equation related to any line is zero ((12) in Chakravorty and Das [10]). This discriminant is defined as the line SI, or the SI of the line’s receiving end node. In this case, the node with the minimum stability index is most sensitive to voltage collapse.

2. The line SI proposed by Moghavvemi and Faruque [9]. This indicator requires reducing the system into a two-bus equivalent model. The system SI $L_p$ is in this case derived from the discriminant of the equation of the received real power in the reduced two-bus network. The critical point occurs when $L_p$ is one ($L_p \leq 1$).

4 The reduced network SI also proposed by Moghavvemi and Faruque [8]. This indicator is also based on the two-bus equivalent model used in case (4). The system SI ((18) in Jasmon and Lee [7]) is $L \leq 1$. The critical point again occurs when $L$ is one.

4 Numerical results

The proposed conic optimisation method for system capability computation was tested on 11 different radial networks gathered from the literature [10, 18–25]. The corresponding stability index (14) was compared with the stability indices discussed in Section 3.2. All programs were coded in MATLAB and the numerical testing was carried out on a Pentium IV, 1.89 GHz PC with 256 Mbytes of RAM.

4.1 Computation of the load capabilities

The implementation of the conic optimisation approach for the analysis of the system load capability was initially validated by comparing with the critical loading conditions of the 69-node test system reported in Chakravorty and Das [10]. In particular, the conic optimiser was used to compute the total real/reactive critical load (TPL/TQL) and the minimum load voltage ($V_{\text{min}}$) for three values of the substation voltage. A comparison of results is shown in Table 1, from which it is evident that the values are in agreement. The conic solution, however, predicts a slightly higher load capability and consequently a lower voltage profile.

In fact, the results in Chakravorty and Das [10] were obtained by gradually increasing the system load vector and running Csendes’ radial load-flow [18] to the point just before voltage collapse occurs. Voltage collapse is said to occur when the load-flow solution diverges. This SLF procedure was implemented in a program in order to verify the conic optimisation results of the other radial systems. In the program, the load variation factor was increased by a step of $\Delta \lambda = 0.001$, thus producing a factor accurate to three digits after the decimal point. This would enable proper validation of the accuracy of the conic solution. Table 2 shows a summary of the results.

<table>
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<tr>
<th>$V_1$, pu</th>
<th>Conic solution</th>
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<tr>
<td></td>
<td>TPL, MW</td>
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<tr>
<td>1.0</td>
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</tr>
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<td>1.025</td>
<td>12.800</td>
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<tr>
<td>1.005</td>
<td>13.487</td>
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Table 1: Load capability of the 69-node system

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Table 2: Load capability computation using conic and sequential load flow approaches

<table>
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<tr>
<th>N</th>
<th>Data from ref.</th>
<th>V₁, pu</th>
<th>Method</th>
<th>λ</th>
<th>TPL, MW</th>
<th>TQL, MVAR</th>
<th>Vₘᵢₙ, pu</th>
<th>Conic iter.</th>
<th>Conic time, s</th>
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<td>25.5364</td>
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</table>

The results were obtained using both conic programming and the SLF approach. (The results in columns 6 and 7 which are marked by an asterisk sign are in pu.) It is evident that for practical purposes, the solutions for λ, TPL, TQL and Vₘᵢₙ are almost identical. The load variation factor is minimum for the 28-node system in Ghosh and Das [22], thus indicating that this system is the closest to voltage collapse. The last two columns in Table 2 show the CPU time and iterations for the conic solution. The conic optimisation time was always around 1 s, with the maximum time being 1.36 s. As the optimiser technology improves, the conic solution time would continue to improve. It should be noted that the simple implementation of the SLF approach with a small step size is used solely for validation of the conic solution results. It cannot be used to draw conclusions about the relative computational efficiencies of the conic and SLF-based approaches.

4.2 Computation of the stability indicators

The SIs in Section 3 were computed for each of the test systems in Table 2. Table 3 shows the SIs at both the base load and the critical load levels. Column 3 in Table 3 shows the stability index in (14). In radial systems, the most vulnerable node is the one having the lowest voltage magnitude. For each of the studied systems, this node is reported in the fourth column in Table 3. Columns 5 through 9 show the voltage SIs of systems, this node is reported in the fourth column in Table 3. The SIs in Section 3 were computed for each of the test systems. The column SI is defined only if V₁ = 1 pu, the cells in column 9 of Table 3 shows the SI in Chakraborty and Das [10]/corresponding line/corresponding sending and receiving end nodes, the results were obtained for 11 radial networks. The results were obtained using both conic programming and the SLF approach. Analysis of the results in Table 3 reveals that when a system load change is considered, the bus SI [1] may not indicate how much additional load can be tolerated before voltage collapse. This conclusion has been reached in Chebbo et al. [1] and is evidenced in Table 3 by the fact that at the critical load, the bus SI is less than one. The bus SI in Chebbo et al. [1] is known to approach unity at the critical load when a single load change is considered. A comparison of the line SIs in Chakraborty and Das [10] and Moghavvemi and Faruque [9] shows that the SI in [10] (column 6) corroborated better with the actual loading condition. In fact, this SI in the vast majority of cases identified the line that is connected to the most vulnerable node, i.e. the node in column 4 of Table 3. The line SI in Moghavvemi and Faruque [9] (column 7) fell short of exhibiting the same level of corroboration. The reduced single line equivalent network indices in columns 8 and 9 were both capable of measuring the relative criticality of the loading condition. (Because the SI in Jasmon and Lee [7] is defined only if V₁ = 1 pu.) As compared to the indicators in columns 5 through 9, the proposed SI (i.e. Lᵥ) in column

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3 yields the most accurate measure of voltage stability. It is always unity at the critical load and thus provides a yardstick against which all loading conditions can be judged. This is not the case for the other SIs which are meaningful only when compared at several loading conditions. As discussed in Section 3.1, the indicator \( L_c \) also provides a measure of how much additional load the distribution system can tolerate before collapse. Such an indication is the most useful for static voltage stability analysis [1].

5 Conclusion

This paper has shown that the radial distribution load capability computation can be formulated as a conic quadratic optimisation problem. Such problems can be solved efficiently using interior-point methods [16]. Numerical results indicate that the load capability can be calculated by a commercial interior-point conic optimiser. A by-product of the conic solution is a static voltage stability index. This index gives a clear measure of the distance to collapse as compared to other previously published indices. Because of high computational speed, this solution method has the potential to be a faster option in SCADA applications. The proposed formulation can be extended to unbalanced three-phase networks.

6 References