Robust output feedback model predictive control of constrained linear systems

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Received 13 August 2005; received in revised form 22 November 2005; accepted 10 March 2006
Available online 4 May 2006

Abstract

This paper provides a solution to the problem of robust output feedback model predictive control of constrained, linear, discrete-time systems in the presence of bounded state and output disturbances. The proposed output feedback controller consists of a simple, stable Luenberger state estimator and a recently developed, robustly stabilizing, tube-based, model predictive controller. The state estimation error is bounded by an invariant set. The tube-based controller ensures that all possible realizations of the state trajectory lie in a simple uncertainty tube the ‘center’ of which is the solution of a nominal (disturbance-free) system and the ‘cross-section’ of which is also invariant. Satisfaction of the state and input constraints for the original system is guaranteed by employing tighter constraint sets for the nominal system. The complexity of the resultant controller is similar to that required for nominal model predictive control.

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Keywords: Output feedback; Robust model predictive control; Observer; Robust control invariant tubes

1. Introduction

Model predictive control has received considerable attention driven largely by its ability to handle hard constraints as well as nonlinearity. An inherent problem is that model predictive control normally requires full knowledge of the state (Findeisen, Imsland, Allgöwer, & Foss, 2003; Mayne, Rawlings, Rao, & Scokaert, 2000). Whereas, in many control problems, not all states can be measured exactly. In practice this problem is often overcome by employing ‘certainty equivalence’. For linear systems not subject to disturbances that employ an observer and linear control, stability of the closed-loop can be guaranteed by the separation principle. However, when state and output disturbances are present and the system or its controller is nonlinear (as is the case with model predictive control of constrained systems), stability of the closed-loop cannot, in general, be ensured by simply combining a stable estimator with a stable state feedback controller (Atassi & Khalil, 1999; Teel & Praly, 1995). Robust stability is obtained if the nominal system is inherently robust and the estimation errors are sufficiently small; however, predictive controllers are not always inherently robust (Grimm, Messina, Teel, & Tuna, 2004; Scokaert, Rawlings, & Meadows, 1997).

An appealing approach for overcoming this drawback is to use robust controller design methods that take the state estimation error directly into account. In the late sixties, Witsenhausen (1968a, 1968b) studied robust control synthesis and set-membership estimation of linear dynamic systems subject to bounded uncertainties; this work was followed by Schweppe (1968), Bertsekas and Rhodes (1971a,1971b), Glover and Schweppe (1971) and by related work on viability (Aubin, 1991; Kurzhanski & Vályi, 1997; Kurzhanski & Filippova, 1993). Further results include: a paper by Blanchini (1990) that deals with linear, constrained, uncertain, discrete-time systems, an anti-windup scheme that employs invariant sets in a similar fashion to their use in this

\textsuperscript{\textasteriskcentered} This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Franco Blanchini under the direction of Editor Roberto Tempo. Research supported by the Engineering and Physical Sciences Research Council, UK.

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0005-1098/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2006.03.005
Several authors, e.g. Bemporad and Garulli (2000) advocate set-membership estimation for model predictive control; to reduce complexity in such controllers, Chisci and Zappa (2002) employs efficient computational methods. For a similar system class, Löfberg (2002) proposes joint state estimation and feedback using min-max optimization. For linear systems with input constraints, the method in Lee and Kouvaritakis (2001) achieves stability by using invariant sets for an augmented system. By constructing an invariant set for the observer error, Kouvaritakis, Wang, and Lee (2000) adapts the model predictive controller in Cannon, Kouvaritakis, Lee, and Brooms (2001) to ensure closed-loop asymptotic stability. In Fukushima and Bitmead (2005) the authors use a comparison system to establish ultimate boundedness of the trajectories of a closed-loop system consisting of an observer and a model predictive controller.

In this paper we consider the output feedback problem for constrained linear discrete-time systems subject to state and measurement disturbances. The basic idea is to consider the state estimation error as an additional, unknown but bounded uncertainty that can be accounted for in a suitably modified robust model predictive controller that controls the nominal observer state rather than the unknown but bounded system state. The state estimation error is bounded by a simple, pre-computed, invariant set. The controller uses a tube, the center of which is obtained by solving a conventional (disturbance-free) model predictive control problem that yields the center of the tube; the tube is obtained by ‘adding’ to its center another simple, pre-computed, invariant set. Tighter constraints in the optimal control problem solved online ensure that all realizations of the state trajectory satisfy the state and control constraints.

In comparison to previous work on *output model predictive control*, simplicity is achieved by using a Luenberger observer (that is less complex than a set-membership state estimator), an invariant set that bounds the estimation error, as in Shamma (2000), and a simple tube-based robust model predictive controller that is nearly as simple to implement as a nominal linear model predictive controller. Robust exponential stability of a “minimal” robust invariant set is established.

**Nomenclature:** In the following \( N \triangleq [0, 1, 2, \ldots] \), \( N_+ \triangleq \{1, 2, \ldots\} \) and \( N_q \triangleq \{0, 1, \ldots, q\} \). A polyhedron is the (convex) intersection of a finite number of open and/or closed half-spaces and a polytope is the closed and bounded polyhedron. Given two sets \( \mathcal{U} \) and \( \mathcal{Y} \), such that \( \mathcal{U} \subseteq \mathbb{R}^n \) and \( \mathcal{Y} \subseteq \mathbb{R}^m \), Minkowski set addition is defined by \( \mathcal{U} \oplus \mathcal{Y} \triangleq \{u + v \mid u \in \mathcal{U}, v \in \mathcal{Y}\} \) and Minkowski (Pontryagin) set difference by \( \mathcal{U} \ominus \mathcal{Y} \triangleq \{x \mid x \oplus \mathcal{Y} \subseteq \mathcal{U}\} \). A set \( \mathcal{U} \subseteq \mathbb{R}^n \) is a C-set if it is a compact, convex set that contains the origin in its (non-empty) interior. Let \( d(z, X) \triangleq \inf \{\|z - x\|_p \mid x \in X\} \) where \( \cdot \|_p \) denotes the \( p \) vector norm. Let \( \rho(A) \) denote spectral radius of a given matrix \( A \in \mathbb{R}^{n \times n} \).

2. Proposed control methodology

We consider the following uncertain discrete-time linear time-invariant system:

\[
x^{i+1} = Ax^{i} + Bu^{i} + w^{i}, \quad y = Cx^{i} + v^{i},
\]

where \( x^{i} \in \mathbb{R}^n \) is the current state, \( u^{i} \in \mathbb{R}^m \) is the current control action, \( x^{i+1} \) is the successor state, \( w^{i} \in \mathbb{R}^n \) is an unknown state disturbance, \( y \in \mathbb{R}^p \) is the current measured output, \( v^{i} \in \mathbb{R}^p \) is an unknown output disturbance, \( (A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \), the couple \( (A, B) \) is assumed to be controllable and the couple \((A, C)\) observable. The state and additive disturbances \( w \) and \( v \) are known only to the extent that they lie, respectively, in the \( C \) sets \( \mathcal{W} \subseteq \mathbb{R}^p \) and \( \mathcal{V} \subseteq \mathbb{R}^p \): \( (w^{i}, v^{i}) \in \mathcal{W} \times \mathcal{V} \). Let \( \bar{y}(i; x, u, w) \) denote the solution of (1) at time \( i \) if the initial state (at time 0) is \( x \) and the control and disturbance sequences are, respectively, \( u \triangleq (u(0), u(1), \ldots) \) and \( w \triangleq (w(0), w(1), \ldots) \). System (1) is subject to the following set of hard state and control constraints:

\[
(x, u) \in X \times U,
\]

where \( X \subseteq \mathbb{R}^n \) and \( U \subseteq \mathbb{R}^m \) are polyhedral and polytopic sets, respectively, and both contain the origin as an interior point.

To estimate the state a simple Luenberger observer is employed (as in Shamma, 2000):

\[
\dot{x} = A\hat{x} + Bu + L(y - \bar{y}), \quad \bar{y} = C\hat{x},
\]

where \( \hat{x} \in \mathbb{R}^n \) is the current observer state, \( u \in \mathbb{R}^m \) is the current control action, \( \bar{x} \) is the successor state of the observer system, \( \bar{y} \in \mathbb{R}^p \) is the current observer output and \( L \in \mathbb{R}^{n \times p} \). The estimated state \( \hat{x} \) satisfies the following uncertain difference equation:

\[
\dot{x} = A\hat{x} + Bu + LC\bar{x} + Lw,
\]

while the state estimation error \( \hat{x} \) satisfies

\[
\dot{x} = A_L\hat{x} + (w - Lv), \quad A_L \triangleq A - LC,
\]

where \( L \in \mathbb{R}^{n \times p} \) satisfies \( \rho(A_L) < 1 \).

We control the observer (3) in such a way that the real state \( x = \hat{x} + \tilde{x} \) always satisfies the state constraint and the associated control always satisfies the control constraint. To achieve this, we introduce a third dynamic system, the nominal system obtained from (1) by neglecting the disturbances \( w \) and \( v \):

\[
\tilde{x} = A\tilde{x} + Bu,
\]

where \( \tilde{x} \in \mathbb{R}^n \) denotes the nominal system state and \( \tilde{u} \in \mathbb{R}^m \) is the input to the nominal system. Let \( \bar{u}(i; \tilde{x}, \tilde{u}) \) denote the state of the nominal system (6) at time \( i \) if its initial state (at time 0) is \( \tilde{x} \) and if the control sequence is \( \bar{u} \triangleq (\bar{u}(0), \bar{u}(1), \ldots) \). Choosing an initial state \( \tilde{x} \) and a nominal control sequence \( \bar{u} \) yields a nominal state sequence \( \tilde{x} \) \( \triangleq \bar{u}(i; \tilde{x}, \tilde{u}) \), \( i = 0, 1, \ldots \); this sequence is the center of a tube that we shall shortly construct.
In order to counteract the disturbances we combine in the control $u$ a feed-forward part, given by the tube based model predictive controller, and a feedback part

$$u = \tilde{u} + Ke, \quad (7)$$

where $e \triangleq \tilde{x} - \hat{x}$ and $K \in \mathbb{R}^{m \times n}$ satisfies $\rho(A + BK) < 1$. With this control, the closed-loop observer state satisfies

$$\dot{x}^+ = A\tilde{x} + B\tilde{u} + BK e + LC\bar{x} + Lv, \quad (8)$$

while the error $e$ between the observer state and the nominal system satisfies the difference equation

$$e^+ = A_K e + (LC\bar{x} + Lv), \quad A_K \triangleq A + BK. \quad (9)$$

We choose an initial state $\tilde{x}(0)$ and a nominal control sequence $\tilde{u}$ to obtain the nominal state sequence $\tilde{x}$. By definition, the observer state differs from the nominal state by $e (\tilde{x} - \hat{x} + e)$, and the actual state differs from the observer state by $\tilde{e} (x - \tilde{x} + \hat{x})$, so that

$$x = \tilde{x} + e + \hat{x}. \quad (10)$$

Providing we can bound $e$ and $\tilde{x}$, we can choose $\tilde{x}(0)$ and $\tilde{u}$ so that the actual (unknown) state and control satisfy the original constraints.

3. Bounding the estimation and control errors

We recall the following standard definitions (Blanchini, 1999; Kolmanovskiy & Gilbert, 1998):

**Definition 1.** A set $\Omega \subset \mathbb{R}^n$ is positively invariant for the system $x^+ = f(x)$ and the constraint set $\%$ if $\Omega \subseteq \%$ and $f(x) \in \Omega$, $\forall x \in \Omega$. A set $\Omega \subset \mathbb{R}^n$ is robustly positively invariant for the system $x^+ = f(x, w)$ and the constraint set $\%$, $\%_w$ if $\Omega \subseteq \%$ and $f(x, w) \in \Omega$, $\forall w \in \%_w$, $\forall x \in \Omega$.

Our next step is to establish that the estimation and control errors (respectively, $\tilde{x}$ and $e$) can be bounded by robust positively (control) invariant sets. The difference equation (5) can be rewritten in the form

$$\dot{x} = A_L \tilde{x} + \delta, \quad \delta \triangleq w - Lv, \quad (11)$$

where the ‘disturbance’ $\delta$ lies in the $C$ set $\tilde{A}$ defined by

$$\tilde{A} \triangleq \% \oplus (-L\%_w). \quad (12)$$

Since $\rho(A_L) < 1$, there exists a $C$ set $\tilde{S}$ that is finite time computable and robust positively invariant (Raković, Kerrigan, Kouramas, & Mayne, 2005; Raković, 2005) for the system (11) and constraint set $(\mathbb{R}^n, \tilde{A})$. The set $\tilde{S}$ satisfies $A_L \tilde{S} \oplus \tilde{A} \subseteq \tilde{S}$. Since $\bar{x}(0) \in \tilde{S}$ implies $\bar{x}(i) \in \tilde{S}$ for all $i \in \mathbb{N}$, an obvious consequence is:

**Proposition 1.** If the initial system and observer states, $x(0)$ and $\hat{x}(0)$, respectively, satisfy $\bar{x}(0) = x(0) - \hat{x}(0) \in \tilde{S}$, then $\bar{x}(i) \in \tilde{S}$ for all $i \in \mathbb{N}$, and all admissible disturbance sequences $w$ and $v$.

The assumption that $\tilde{x}(0) \in \tilde{S}$ is a steady state assumption for the observer error. If the observer state $\hat{x}$ lies in the tightened constraint set $\% \ominus \tilde{S}$ the original state $x$ is guaranteed to lie in $\%$. We bound the control error $e$ by an invariant set. The difference equation (9) can be rewritten in the form

$$e^+ = A_K e + \tilde{\delta}, \quad \tilde{\delta} \triangleq LC\bar{x} + Lv, \quad (13)$$

where (since $\tilde{x}$ is bounded by $\tilde{S}$) the “artificial disturbance” $\tilde{\delta}$ lies in the set $\tilde{A}$ defined by

$$\tilde{A} \triangleq LC\tilde{S} \oplus L\%_v. \quad (14)$$

Because $\rho(A_K) < 1$, there exists a $C$ set $\tilde{S}$ that is finite time computable and robust positively invariant (Raković et al., 2005; Raković, 2005) for system (13) and constraint set $(\mathbb{R}^n, \tilde{A})$. The set $\tilde{S}$ satisfies $A_K \tilde{S} \oplus \tilde{A} \subseteq \tilde{S}$. Since $e(0) \in \tilde{S}$ implies $e(i) \in \tilde{S}$ for all $i \in \mathbb{N}$, an obvious consequence is:

**Proposition 2.** If the initial observer and nominal system states, $\tilde{x}(0)$ and $\tilde{\tilde{x}}(0)$, respectively, satisfy $e(0) = \tilde{x}(0) - \tilde{\tilde{x}}(0) \in \tilde{S}$, then $\tilde{x}(i) \in \tilde{\tilde{x}}(i) \ominus \tilde{S}$ for all $i \in \mathbb{N}$, all admissible disturbance sequences $w$ and $v$.

Defining $\tilde{S} \ominus \tilde{S}$, we obtain:

**Proposition 3.** Suppose $\tilde{x}(0), \tilde{x}(0)$ and $\tilde{u}(0), \tilde{u}(0), \ldots$ are given. If the initial system, observer, and nominal system states satisfy $\bar{x}(0) = x(0) - \hat{x}(0) \in \tilde{S}$ and $e(0) = \tilde{x}(0) - \tilde{\tilde{x}}(0) \in \tilde{S}$, and $u(i) = \tilde{u}(i) + (K(\tilde{x}(i) - \tilde{\tilde{x}}(i)))$ for all $i \in \mathbb{N}$, then the system states satisfy $\bar{x}(i) \in \bar{x}(i) \ominus \tilde{S}$ for all $i \in \mathbb{N}$ and all admissible disturbance sequences $w$ and $v$ $(\tilde{x}(i) \ominus \tilde{\tilde{x}}(i), \tilde{u}(i))$.

The control constraint $u \in U$ induces a constraint on $\tilde{u}$ via the control law $u = \tilde{u} + Ke$; to ensure $u \in U$, $\tilde{u}$ must satisfy the tighter constraints $\tilde{u} \in U \ominus K\tilde{S}$. In order to ensure that the unknown state satisfies state constraints $x \in \%$ we must ensure that $\tilde{x} \in \% \ominus \tilde{S}$ and that $\bar{x} \in \% \ominus (\tilde{S} \ominus \tilde{S})$. Summarizing, to employ our procedure, we assume:

**A1.** There exist a controller $K$ and an output injection $L$ such that $\rho(A_L) < 1$, $A_K = A + BK$ and $\rho(L) < 1$, $A_L = L + LC$ and the associated robust positively invariant sets $\tilde{S}$ and $\tilde{S}$ satisfy $S = \tilde{S} \ominus \tilde{S} \subseteq \%$ and $K\tilde{S} \subseteq U$.

The assumptions on $\tilde{S}$ and $\bar{x}$, which are satisfied if $\%\%$ and $\%\%$ are sufficiently small, can be verified by solving a linear program as shown in Raković et al. (2005), Raković (2005), robustness to large uncertainty cannot necessarily be achieved. Combining Propositions 2 and 3 and the discussion above, we obtain:

**Theorem 1.** (i) Suppose the initial system, observer, and nominal system states all lie in $\%$ and satisfy $\bar{x}(0) = x(0) - \hat{x}(0) \in \tilde{S}$ and $e(0) = \bar{x}(0) - \tilde{x}(0) \in \tilde{S}$. Then (i) $x(0) \in \tilde{S}$ and $\bar{x}$. If, in addition, initial state $\bar{x}(0)$ and control sequence $\tilde{u}$ of the nominal system satisfy the tighter constraints $\bar{x}(i) = \bar{x}(i) \ominus \tilde{S}$ and $\tilde{u}(i) \in \% \ominus \%$ and $\tilde{u}(i) = \tilde{u}(i) \ominus \% \ominus K\tilde{S}$ for all $i \in \mathbb{N}$, then (ii) the state...
4. Robust output feedback MPC

Theorem 1 provides the ingredients we require to obtain output model predictive control of the original system by employing the model predictive controller in Mayne, Seron, and Raković (2005) to control robustly the observer system (4):

\[ \dot{x}^+ = A\hat{x} + Bu + \delta, \quad \hat{\delta} \triangleq LC\hat{x} + Lw, \]

where \( \hat{\delta} \in \Delta = LC\hat{S} \oplus L\mathbb{V} \). Let the cost \( V_N(\hat{x}, \hat{u}) \) be defined by

\[ V_N(\hat{x}, \hat{u}) = \sum_{i=0}^{N-1} (\ell(x(i), \hat{u}(i)) + V_f(\hat{x}(N))), \]

where \( N \) is the horizon and \( V_f(\cdot) \) is the terminal cost function and \( \ell(\cdot) \) is the stage cost defined by

\[ \ell(x, u) = (1/2)[x'Qx + u'Ru], \quad V_f(x) = (1/2)x'P_\infty x, \]

where \( P, Q, R \) are assumed to be positive definite. Note that the results in this paper are also applicable to the case when \( \ell(\cdot) \) and \( V_f(\cdot) \) are defined using polytopic vector norms (e.g. \( \ell_1, \ell_\infty \)). The state of the nominal system \( \hat{x} \) and control applied to the nominal system \( \tilde{u} \) are subject to the tight constraints defined by:

\[ (\tilde{u}, \tilde{x}) \in \tilde{X} \times \tilde{U}, \quad \tilde{X} \triangleq \mathcal{X} \oplus \mathcal{S}, \quad \tilde{U} \triangleq \mathcal{U} \cup \mathcal{K}\hat{S}. \]

The initial and final states \( \hat{x} \) and \( \hat{x}(N) \) of the nominal system are required to satisfy, respectively,

\[ \hat{x} \in \tilde{X} \oplus \tilde{S}, \quad \hat{x}(N) \in X_f. \]

The set \( X_f \) is the terminal constraint set for the nominal system. The initial nominal state \( \tilde{x} \) is, with the nominal control sequence, \( u \), a decision variable; given \( \tilde{x} \), the set of admissible control sequences is

\[ \mathcal{U}_N(\tilde{x}) = \{ \tilde{u} | \tilde{u}(i) \in \tilde{U}, \quad \tilde{\phi}(i; \tilde{x}, \tilde{u}) \in \tilde{X}, \quad \forall i \in \mathbb{N}_{N-1}, \quad \tilde{\phi}(N; \tilde{x}, \tilde{u}) \in X_f \}. \]

Let \( \tilde{X}_N \triangleq \{ \tilde{x} | \mathcal{U}_N(\tilde{x}) \neq \emptyset \} \). The nominal optimal control problem \( P_N(\tilde{x}) \), solved on-line is:

\[ V^*_N(\tilde{x}) = \min_{\tilde{x}, \tilde{u}} \{ V_N(\tilde{x}, \tilde{u}) | \tilde{u} \in \mathcal{U}_N(\tilde{x}), \quad \tilde{x} \in \tilde{X} \oplus \tilde{S} \}, \]

where \( K_f \) is not necessarily the same as \( K \). The solution of \( P_N(\tilde{x}) \) is \((\tilde{x}^*(\tilde{x}), \tilde{u}^*(\tilde{x})) \) from which, as shown in Mayne et al. (2005), the model predictive control law \( \kappa_N(\cdot) \) is obtained

\[ \kappa_N(\tilde{x}) \triangleq \tilde{u}^*_N(\tilde{x}) + K(\tilde{x} - \tilde{x}^*(\tilde{x})), \]

where \( \tilde{u}^*_N(\tilde{x}) \) is the first element in the sequence \( \tilde{u}^*(\tilde{x}) \).

Definition 2. A set \( \mathcal{R} \) is robustly exponentially stable (Lyapunov stable and exponentially attractive) for the system \( x^+ = f(x, k(x), w), w \in \mathbb{W} \), with a region of attraction \( \mathcal{S} \) if there exists a \( c > 0 \) and a \( \gamma \in (0, 1) \) such that any solution \( x(\cdot) \) of \( x^+ = f(x, k(x), w) \) with initial state \( x(0) \in \mathcal{S} \), and admissible disturbance sequence \( w(\cdot) \) (\( w(i) \in \mathbb{W} \) for all \( i \in \mathbb{N} \)) satisfies

\[ d(x(i), \mathcal{R}) \leq c^i d(x(0), \mathcal{R}) \]

for all \( i \in \mathbb{N} \).

We can now establish our second main result:

Theorem 2. Suppose \( \tilde{X}_N \) is bounded. Then, (i) the set \( \tilde{S} \times \tilde{S} \) is robustly exponentially stable for the composite system \( \tilde{x}^+ = A\tilde{x} + BK_N(\tilde{x}) + \tilde{\delta}, \quad \tilde{x}^+ = A\tilde{x} + \hat{\delta} \) with a region of attraction \( \tilde{X}_N \times \tilde{S} \) where \( \tilde{X}_N \triangleq \tilde{X} \oplus \tilde{S} \) and, (ii) any state \( x(0) \in \tilde{x}(0) + \tilde{x}(0) \) where \( (\tilde{x}(0), \tilde{x}(0)) \in \tilde{X}_N \times \tilde{S} \) is robustly steered to \( \mathcal{S} \) exponentially fast while satisfying the state and control constraints.

Proof. (i) Since \( \tilde{x}(0) \in \tilde{S} \), it follows from Proposition 1 that \( \tilde{x}(i) \in \tilde{S} \) for all \( i \in \mathbb{N} \). It follows that the uncertain observer system (15) is identical to the uncertain system considered in Mayne et al. (2005) with \( \tilde{x} \) replacing \( x, \hat{\delta} \) replacing \( u, \) and \( \tilde{A} \) replacing \( W, \) and that problem \( P_N(\hat{x}) \) is, with these substitutions, identical to problem \( P_N(\tilde{x}) \) considered in Mayne et al. (2005). It follows from Theorem 1 in Mayne et al. (2005) that the set \( \tilde{S} \) is robustly exponentially stable for the controlled (uncertain) observer system \( \tilde{x}^+ = A\tilde{x} + BK_N(\tilde{x}) + \hat{\delta} \) which establishes (i). Part (ii) of the theorem follows from the fact that \( x(i) \in \tilde{x}(i) \oplus \tilde{S} \) for all \( i \in \mathbb{N} \).

Note that, from Mayne et al. (2005) and Theorem 2, \( x^+(i)) \to 0, d(x(i)), S \to 0 \) robustly and exponentially fast as \( i \to \infty \). For each \( j \), let \( \tilde{X}_j \equiv \{ x | \mathcal{W}_j(\tilde{x}) \neq \emptyset \} \) where \( \mathcal{W}_j(\tilde{x}) \) is defined by (20), \( j \) replacing \( N \). If A2 holds, the set sequences \( \{ \tilde{X}_j \}, \{ \tilde{X}_j \} \) and \( \{ \tilde{X}_j \} \) are monotonically non-decreasing, i.e. \( \tilde{X}_j \subseteq \tilde{X}_{j+1} \) for all \( j \in \mathbb{N} \). This guarantees the recursive feasibility of the problem \( P_N(\tilde{x}) \). Problem \( P_N(\tilde{x}) \) is a quadratic program only marginally more complex than that required for standard linear model predictive control if the sets \( \tilde{S}, \tilde{S}, \) and \( X_f \), which can be precomputed, are polytopic.

5. Brief example

Our illustrative example is a double integrator:

\[ x^+ = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

with additive disturbances \((w, v) \in \mathbb{W} \times \mathbb{V} \) where \( \mathbb{W} \equiv \{ w \in \mathbb{R}^2 | |w|_\infty \leq 0.1 \} \) and \( \mathbb{V} \equiv \{ u \in \mathbb{R} | |u| \leq 0.05 \} \). The state and
control constraints are \((x, u) \in X \times U\) where \(X = \{x \in \mathbb{R}^n \mid x^T \mathbf{L} x \leq 3\}\) and \(U = \{u \in \mathbb{R} \mid |u| \leq 3\}\) \((x^i)\) is the \(i\)th coordinate of a vector \(x\). The control matrix \(K\) and the output injection matrix \(L\) are \(K = [1 1], L = [1 1]^{T}\). The cost function is defined by \((16)-(17)\) with \(Q = I, R = 0.01\); the terminal value \(V_f(x)\) is the value function \(1/2 x' P f x\) for the unconstrained optimal control problem for the nominal system \(\dot{x}^T = A \dot{x} + B u\) and \(u = K f x\) is the associated LQR controller. The robust positively invariant sets \(\bar{S}, \tilde{S}\) and \(S\) are precomputed using results in Raković et al. (2005); Raković (2005). The terminal constraint set \(\tilde{X}_j\) satisfies \(A2\) and \(A3\) and is the maximal positively invariant set (Blanchini, 1999; Gilbert & Tan, 1991) for system \(\dot{x}^T = (A + BK) \bar{x}\) under the tighter constraints \(\bar{X} = \bar{X}_0 \Sigma_{j=1}^{N} S\) and \(\bar{U} = U \oplus K \bar{S}\). The horizon is \(N = 13\). The sets \(\tilde{X}_j\) expand into the bottom left quadrant of \(\mathbb{R}^2\) as \(j\) increases. The tube \(\{\bar{x}^*(\tilde{x}(i)) \oplus S\}\) that bounds the unknown state trajectory, for a random initial state \(x_0 \in \bar{X}_0 \oplus \bar{S}\) where \(\tilde{x}_0 = (-3, -8)^T\) and a random sequence of extreme admissible state and output disturbances, is shown in Fig. 1.

6. Conclusions

This paper presents a simple output feedback model predictive controller for constrained linear systems with input and output disturbances. Simplicity is achieved by using a Luenberger observer (that is less complex than a set-membership state estimator), an invariant set that bounds the estimation error and a tube-based robust model predictive controller that is almost as simple to implement as a nominal linear model predictive controller. Robust exponential stability is established.

References


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