Robust stabilisation of multivariable systems under co-prime factor perturbations: Directionality and Super-optimisation

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Abstract—Robust stabilisation of MIMO LTI systems under normalised co-prime factor unstructured uncertainty is considered. The maximal robust stability radius \( \epsilon^* \) is derived and a “worst-case” direction is identified, along which all boundary uniformly-destabilising perturbations are shown to lie, i.e. all perturbations of norm \( \epsilon^* \) which destabilise the closed-loop system for every optimal (maximally robust) controller. By imposing a parametric constraint on the projection of admissible perturbations along this direction (uniformly in frequency), it is shown that it is possible to extend the robust stability radius in every other direction, using a subset of all optimal (maximally-robust) controllers, by solving a super-optimal Nehari extension problem. A closed-form expression is obtained for the constrained robust stability radius, \( \mu^*(\delta) \) which depends on the first two-superoptimal levels of the closed-loop system, while the identified “worst-case” direction corresponds to the maximal Schmidt pair of a Hankel operator related to the problem. The paper extends the results of [GHJ00] for additive uncertainty models to the co-prime uncertainty case.

I. NOTATION

A. General

All systems considered in the paper are assumed linear, time invariant and finite-dimensional. Let \( \mathcal{R}^{p \times m} \) denote the space of proper \( p \times m \) rational matrix functions in \( s \) with real coefficients. Associated with \( P(s) \in \mathcal{R}^{p \times m} \) of McMillan degree \( n \) is a state-space realisation:

\[
P(s) = C(sI - A)^{-1}B + D
\]

where \( A \in \mathcal{R}^{n \times n} \), \( B \in \mathcal{R}^{n \times m} \), \( C \in \mathcal{R}^{p \times n} \) and \( D \in \mathcal{R}^{p \times m} \). For \( P(s) \in \mathcal{R}(s)^{p \times m} \) let \( \tilde{P}^{\prime}(-\pi) \) denote the para-hermitian conjugate of \( P(s) \). Let \( P(s) \) be partitioned as \( P_{ij}(s) \), \( i, j = 1, 2 \). Then a state space realisation of \( \tilde{P}(s) \) can be written as:

\[
\tilde{P}(s) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}
\]

and

\[
P_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}
\]

is a state-space realisation of \( P_{ij}(s) \). A lower linear fractional transformation of \( P(s) \) and \( K(s) \) is defined as

\[
\mathcal{F}_L(P, K) = P_{11} + P_{12}(I - P_{22}K)^{-1}P_{21}
\]

where \( K \) is of dimension \( m \times p \) if \( P_{22}(s) \) has dimension \( p \times m \) and the indicated inverse exists. Similarly we define the upper linear fractional transformation of \( P(s) \) and \( K(s) \) as:

\[
\mathcal{F}_u(P, K) = P_{22} + P_{21}(I - P_{11}K)^{-1}P_{12}
\]

for a compatible partitioning of \( P(s) \) with \( K(s) \) and provided that the indicated inverse exists.

The factorisation

\[
P(s) = N(s)M^{-1}(s) = M^{-\tilde{I}}\tilde{N}(s)
\]

is said to be both right and left co-prime (lef and rcf) if it satisfies the following Diophantine matrix equation, known as the Bezout identity:

\[
\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = I
\]

for appropriate functions \( U, V, \tilde{U}, \tilde{V} \in \mathcal{RH}_\infty \). Further, if

\[
M\tilde{M}^\sim + \tilde{N}\tilde{N}^\sim = I, \quad M^\sim M + N^\sim N = I
\]

then the lef and rcf factorisations are normalised.

The space \( \mathcal{RH}_\infty \) consists of all proper real-rational transfer matrix functions which are analytic on the imaginary axis. \( \mathcal{RH}_\infty^+ \) and \( \mathcal{RH}_\infty^- \) are the subspaces of \( \mathcal{RH}_\infty \) consisting of all real-rational proper matrix functions which are analytic in the closed right-half plane and closed left-half plane, respectively. Thus \( \mathcal{RH}_\infty = \mathcal{RH}_\infty^+ \oplus \mathcal{RH}_\infty^- \) where \( \oplus \) denotes direct sum of subspaces. The norm \( \| \cdot \|_\infty \) denotes either the \( L_\infty \)-norm of a function \( \in \mathcal{L}_\infty \) or the \( H_\infty \)-norm of a function \( \in \mathcal{H}_\infty \), depending on context.

If \( \Gamma \) is an operator, then \( \| \Gamma \| \) denotes its induced norm. Here we make use the Hankel operator with symbol \( G \in \mathcal{L}_\infty \), defined as [Fra87]:

\[
\Gamma_G : H_2 \rightarrow H_2^\perp, \quad \Gamma_G = \Pi - \Lambda_G|H_2
\]

where \( \Lambda_G \) denotes the Laurent operator with symbol \( G \): \( \Lambda_G f = G f \) for \( f \in \mathcal{L}_2 \) and \( \Pi_- \) denotes orthogonal projection \( \mathcal{L}_2 \rightarrow H_2^\perp \). Note that \( \Gamma_G = 0 \) if \( G \in \mathcal{H}_\infty \). An equivalent “time-domain” definition of the Hankel operator is given as follows. Let \( G \in \mathcal{H}_\infty \) have a state-space realisation \( \mathcal{L}_2 = (A, B, C, 0) \) with \( \lambda(A) \subset \mathcal{C}_+ \). The inverse bilateral transform of \( G \) is

\[
g(t) = -Ce^{At}B \text{ for } t < 0, \quad g(t) = 0 \text{ for } t \geq 0
\]
The time-domain analogue of $\Gamma_G$, $\Gamma_g$, maps a function $u \in L_2[0, \infty)$ to a function $y \in L_2(-\infty, 0]$ defined by:
\[
y(t) = \int_0^\infty g(t - \tau)u(\tau)d\tau = -Ce^{At}\int_0^\infty e^{-At}Bu(\tau)d\tau, \ t < 0
\]
Note that $\Gamma_g = \Psi_o \Psi_e$, where $\Psi_o$ and $\Psi_e$ are the observability and controllability operators, respectively, defined as:
\[
\Psi_o : \mathbb{R}^n \rightarrow L_2(-\infty, 0], \ (\Psi_0 x)(t) = Ce^{At}x, \ t < 0
\]
and
\[
\Psi_e : L_2[0, \infty) \rightarrow \mathbb{R}^n, \ \Psi_e u = -\int_0^\infty e^{-At}Bu(\tau)d\tau
\]
where $n = \dim(A)$ and $L_2(-\infty, 0]$, $L_2(0, \infty]$ denotes the space of square-integrable functions with support on the indicated intervals. The Hankel singular values of $\Gamma_G$ (or $\Gamma_g$) are defined as the square-roots of the non-zero eigenvalues of $\Gamma_G^*\Gamma_G$, denoted as $\sigma_i^2$ and indexed in non-increasing order of magnitude. We also write $\sigma_j = \sigma_1(\Gamma_G) = \|G\|_H$. Here $\Gamma_G^*$ denotes the adjoint operator of $\Gamma_G$, $\Gamma_G^* : H_2^\infty \rightarrow H_2$, $\Gamma_G^* = \Pi_+ \Lambda_+^j |H_2^\infty$, where $\Pi_+$ is the orthogonal projection $L_2 \rightarrow H_2$. Assume that $\sigma_j^2$ is a simple eigenvalue of $\Gamma_G^*\Gamma_G$. Then the pair $(g, h)$, where $g$ is the eigenvector of $\Gamma_G^*\Gamma_G$ and $h$ is the eigenvector of $\Gamma_G^*\Gamma_G$ corresponding to $\sigma_j^2$, respectively, is called the maximal Schmidt pair of $\Gamma_G$.

A square matrix function $G \in \mathbb{R}L_\infty$ is called $\gamma$-allpass if $GG^\gamma = G^\gamma G = \gamma^2 I$. A square all-pass function with $\gamma = 1$ is called inner if it lies in $\mathbb{R}H_\infty^\infty$ and anti-inner if it lies in $\mathbb{R}H_\infty^\infty$. $\gamma B H_\infty^{p \times m}$ denotes the set of all $\gamma$-contractions of $H_\infty^{p \times m}$, i.e. all $Q \in H_\infty^{p \times m}$ such that $\|Q\|_\infty \leq \gamma$.

**B. Super-optimisation**

Given $T \in \mathbb{R}L_\infty^{p \times m}$ define
\[
s_i^\infty(T) := \sup_{\omega \in \mathbb{R}} \sigma_i[T(j\omega)], \quad i = 1, 2, \ldots, \min(p, m)
\]
where $\sigma_i[T(j\omega)]$ denotes the $i$-th singular value of $T(j\omega)$, with $\sigma_i \geq \sigma_{i+1}$ for all $i$. Given $R \in \mathbb{R}H_\infty^{-p \times m}$, define recursively the $i$-th super-optimal level of $R$ as:
\[
s_i(R) := \inf_{Q \in S_{i-1}(R)} s_i^\infty(R + Q)
\]
where $i \leq \min(p, m)$ and
\[
S_i(R) := \{Q \in S_{i-1} : s_i^\infty(R + Q) = s_i(R)\}
\]
is the set of all $i$-th super-optimal approximations of $R$ with $S_0(R) = \mathbb{R}H_\infty^{p \times m}$. Clearly $s_1(R) = \|R\|_\infty$, $S_1(R)$ is the set of all Nehari extensions of $R$ and $S_i(R)$ is the set of all $Q \in \mathbb{R}H_\infty^{-p \times m}$ which minimise the sequence $(s_i^\infty(R), s_{i+1}^\infty(R), \ldots, s_{\min(p, m)}^\infty(R))$ with respect to lexicographic ordering. For more general discussion see [JL93], [You86].

**II. Introduction**

In this paper we consider a maximally robust stabilisation problem under normalised coprime-factor uncertainty. We investigate the set of optimal solutions and show that, in the multi-variable case, super-optimisation can be used to guarantee stabilisation of a larger class of perturbations relative to an arbitrary optimal solution. This observation has also been made by Nyman [Nym99], who identified the extended set of perturbations stabilised by the (unique) super-optimal controller. His description of this set, however, is rather implicit (as it is formulated in the form of a weighted-norm) and thus not really suitable for the further investigation of its structure, or for control design purposes. Here, we attempt to identify the stronger robust-stability properties arising by using the super-optimal solution to the problem in terms of directionality. This arises naturally from the observation that every boundary (maximum-norm) perturbation which is uniformly destabilising (i.e. which destabalis the closed-loop system for every maximal robust controller) lies in a certain direction which is identified. Our approach leads to a simple closed-form expression for the constrained robust-stability radius (for every direction other than the worst-case direction) and can be easily applied to control design problems involving both structured and unstructured uncertainty models.

The control setup we are interested in is described in figure 1. Consider the following generalised plant:
\[
P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} := \begin{pmatrix} 0 & I \\ M^{-1} & G \end{pmatrix}
\]
where $P_{22} := G$ admits normalised lcf and rcf:
\[
G = NM^{-1} = \tilde{M}^{-1}\tilde{N}
\]
respectively, as defined by equations (2) and (3). A perturbation on the nominal plant $G$ is said to be permissible if it can be written: $\Delta := (\Delta_N, \Delta_M) \in D_\epsilon$ where
\[
D_\epsilon := \{\Delta : \Delta \in \mathbb{R}H_\infty, \ \|\Delta\|_\infty < \epsilon\}
\]
Then it is easy to verify that the corresponding perturbed plant $G_\Delta$ is [MG90], [ZDG96], [GL95].
\[
F_n(P, \Delta) = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)
\]

Fig. 1. Closed-loop system under stable perturbations

\[
\begin{array}{c}
\Delta_N \\
\Delta_M \\
N \\
K \\
\tilde{M}^{-1} \\
\Gamma \\
\end{array}
\]
as shown in figure 1.

Definition 2.1: Consider the (nominal) feedback system \((G, K)\) in Figure 1 with \(\Delta = 0\). If \((G, K)\) is internally stable we say that \(K\) stabilises \(G\) and write \(K \in \mathcal{K}\) and \((G, K) \in \mathcal{S}\). \((G, K)\) is said to be \(\epsilon\)-robustly stable if and only if \((G_\Delta, K) \in \mathcal{S}\) for every \(\Delta \in \mathcal{T}_\epsilon\).

Theorem 2.1: [Vid85], [MG90] Let \(G \in \mathcal{RL}_\infty\) admit left and right coprime factorisations \(G = NM^{-1} = \tilde{M}^{-1} \tilde{N}\), respectively. Then \((G, K)\) is \(\epsilon\)-robustly stable if and only if \((G, K) \in \mathcal{S}\) and \(\|T\|_\infty \leq \epsilon^{-1}\) where

\[
T = \left(\begin{array}{c}
K \\
I \\
\end{array}\right) (I - GK)^{-1} \tilde{M}^{-1}
\]

It follows immediately that the robust stability radius \(\epsilon^*\) is maximised by solving:

\[
\epsilon^* = \left(\inf_{K \in \mathcal{S}} \|T\|_\infty\right)^{-1}
\]

If a controller \(K\) stabilises \(G\) then it is well-known that it can be written in the bilinear form \(K = (U + MQ)(V + NQ)^{-1}\), where \(Q \in \mathcal{H}_\infty\) and all other terms are defined as in equation (2). Then,

\[
T = \left(\begin{array}{c}
K \\
I \\
\end{array}\right) (I - GK)^{-1} \tilde{M}^{-1}
= \left(\begin{array}{c}
K \\
I \\
\end{array}\right) (V + NQ) = \left(\begin{array}{c}
U + MQ \\
V + NQ \\
\end{array}\right)
= \left(\begin{array}{c}
U \\
V \\
\end{array}\right) + \left(\begin{array}{c}
M \\
N \\
\end{array}\right) Q
\]

Following the standard procedure developed in [Fra87], [Glo86] the maximum robust stabilisation problem can be reduced to a Nehari approximation, by considering the following sequence of norm preserving transformations:

\[
\|T\|_\infty = \left\| \left(\begin{array}{c}
U \\
V \\
\end{array}\right) + \left(\begin{array}{c}
M \\
N \\
\end{array}\right) Q \right\|_\infty =: \|T_1 + T_2 Q\|_\infty
= \|T_1 + (T_2 \quad T_\perp) \left(\begin{array}{c}
Q \\
0 \\
\end{array}\right)\|_\infty
= \|\left(\begin{array}{c}
T_2 \\
T_\perp \\
\end{array}\right) T_1 + \left(\begin{array}{c}
Q \\
0 \\
\end{array}\right)\|_\infty
\]

where we have defined \(T_2 = \left(\begin{array}{c}
M \\
N \\
\end{array}\right), T_\perp := \left(\begin{array}{c}
\tilde{N}^* \\
\tilde{M}^* \\
\end{array}\right)\) and have used equation (2). Thus:

\[
\|T\|_\infty = \left\| \left(\begin{array}{c}
M^* \\
\tilde{N} \\
\tilde{M} \\
\end{array}\right) \left(\begin{array}{c}
U \\
V \\
0 \\
\end{array}\right) + \left(\begin{array}{c}
Q \\
0 \\
\end{array}\right)\right\|_\infty =: \gamma
\]

so that

\[
\|M^* U + N^* V + Q\|_\infty = \sqrt{\gamma^2 - 1}
\]

and

\[
g_{opt} := \inf_{Q \in \mathcal{H}_\infty} \|T\|_\infty = \sqrt{\|M^* U + N^* V\|_H^2 + 1} = \left(\epsilon^*\right)^{-1}
\]

using the Nehari theorem [Glo84].

Hence, the computation of the maximal stability radius involved in the co-prime factor robust stabilisation problem reduces to a Nehari-extension problem. A state-space parametrisation of all optimal solutions to this problem is well-known (see [Glo84]). This solution proceeds via the derivation of a state-space realisation of \(R := M^* U + N^* V\).

Lemma 2.1: [MG90] Assume a \(G \in \mathcal{RL}_\infty\) with a state-space realisation:

\[
G = \left[\begin{array}{c}
A \\
B \\
C \\
\end{array}\right]
\]

Then \(G\) has essentially unique normalised right and left co-prime factor representation \(G = NM^{-1} = \tilde{M}^{-1} \tilde{N}\) (i.e., unique up to multiplication by unitary matrices from the right and left, respectively). Further,

\[
\left(\begin{array}{c}
M \\
N \\
\end{array}\right) \left(\begin{array}{c}
U \\
V \\
\end{array}\right) = \left[\begin{array}{c}
A - BB^* X \\
B^* X \\
C \\
\end{array}\right]
\]

and

\[
\left(\begin{array}{c}
\tilde{V} \\
\tilde{N} \\
\end{array}\right) = \left[\begin{array}{c}
A - ZC^* C \\
B^* X \\
C \\
\end{array}\right]
\]

where \(X\) and \(Z\) are the unique stabilising solutions of the algebraic Riccati equations:

\[
A^* X + X A - XBB^* X + C^* C = 0
\]

and

\[
AZ + ZA^* - ZC^* C Z + BB^* = 0
\]

respectively.

Straight substitution from the lemma above shows that:

\[
R \triangleq \left[\begin{array}{c}
-A^* + XBB^* \\
B^* X \\
0 \\
\end{array}\right] (I + XZ) C^*
\]

after using an appropriate similarity transformation to remove the unobservable part. Thus \(R \in \mathcal{H}_\infty\). The following Theorem can also be easily established:

Theorem 2.2: A controller \(K\) stabilises \(G = NM^{-1} = \tilde{M}^{-1} \tilde{N}\) and satisfies

\[
\left\| \left(\begin{array}{c}
K \\
I \\
\end{array}\right) (I - GK)^{-1} \tilde{M}^{-1}\right\|_\infty \leq \gamma
\]

if and only if either condition 1 or 2 below holds:

1) \(\|R\|_H \leq \sqrt{\gamma^2 - 1}\)

2) \(\left\| \tilde{M}^* \right\|_\infty \leq \sqrt{1 - \gamma^{-2}}\)

Proof: The equivalence with condition 2 is proved in [MG90]. Then using norm preserving transformations the first claim can also be proved.
III. MAIN RESULTS

A. Optimal and Super-optimal approximations

As shown in the previous section, the set of all internally stabilising controllers $\mathcal{K}$ and the corresponding closed-loop systems $T$ are parameterised as:

$$\mathcal{K} = \{(U + MQ)(V + NQ)^{-1} : Q \in \mathcal{H}_\infty\}$$

and

$$T = \left\{ \begin{pmatrix} U & M \\ V & N \end{pmatrix} Q : Q \in \mathcal{H}_\infty \right\}$$

respectively. To parameterise the set of all optimal (maximally robust) controllers $\mathcal{K}_1 \subseteq \mathcal{K}$ and the corresponding set of optimal closed-loop systems $\mathcal{T}_1 \subseteq T$, we need to solve the Nehari extension problem defined in equation (4). The set of all optimal solutions is parameterised in the following theorem. To simplify notation it is assumed that the largest singular value of $\Gamma_R$ is simple (non-repeated).

**Theorem 3.1 (Optimal Nehari approximation):** Consider $R \in \mathcal{RH}_\infty^{p \times m}$ with realisation $(A_R, B_R, C_R, 0)$ defined in equation (6) with $\Lambda(A_R) \subseteq \mathcal{C}_+$. Then there exists $Q_a \in \mathcal{RH}_\infty^{p+m \times (p+m)}$ such that all $Q \in \mathcal{H}_\infty^{p \times m}$ such that $\|R + Q\|_\infty = \|\Gamma_R\| = s_1$ (Nehari optimal approximations of $R$) are given by

$$Q = \mathcal{F}_1(Q_a, s_1^{-1} \mathcal{BH}_\infty^{(p-1) \times (m-1)})$$

The corresponding “error” system is given by

$$H := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} R + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$  \hspace{1cm} (7)

where $\|H_{22}\|_\infty < s_1$. Further, $HH^\sim = H^\sim H = s_1^{-1}I_{p+m-1}$.  

**Proof:** See [Glo84]; see also [JL93] for a more general setting. \hspace{1cm} ■

It follows from Theorem 3.1 that

$$\mathcal{K}_1 = \{(U + MQ)(V + NQ)^{-1} : Q \in \mathcal{F}_1(Q_a, s_1^{-1} \mathcal{BH}_\infty)\}$$

and

$$\mathcal{T}_1 = \left\{ \begin{pmatrix} U & M \\ V & N \end{pmatrix} Q : Q \in \mathcal{F}_1(Q_a, s_1^{-1} \mathcal{BH}_\infty) \right\}$$  \hspace{1cm} (8)

A more revealing parametrisation of $\mathcal{T}_1$ for our purposes can be obtained via the method used to construct super-optimal approximations. Before stating this parametrisation we need the following result.

**Theorem 3.2:** [GHJ00] (i) There exist square inner matrix functions:

$$V = \begin{pmatrix} v & V_\perp \end{pmatrix} \quad \text{and} \quad W^\sim = \begin{pmatrix} W_1^\sim & W_\perp^\sim \end{pmatrix}$$

such that

$$\begin{pmatrix} V^\sim & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} s_1 \alpha(s) & 0 \\ 0 & \hat{H} \end{pmatrix}$$

where $\alpha(s)$ is scalar anti-inner. (ii) $\hat{H}$ can be decomposed as

$$\hat{H} = \hat{R}_a + \hat{Q}_a := \begin{pmatrix} \hat{R} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{pmatrix}$$

where $\hat{R} \in \mathcal{RH}_\infty^{(p-1) \times (m-1)}$ and $\hat{Q}_{ij} \in \mathcal{RH}_\infty$.

Remark: The construction of Theorem 3.2 is an intermediate step for solving the the superoptimal Nehari extension problem using a state-space approach (see [JL93]). It may be shown that the two vectors $v$ and $w$ defined in the Theorem can be obtained via frequency-scaling from the maximal Schmidt vector pair of $\Gamma_R$. \hspace{1cm} ■

**Theorem 3.3:** (i) The set of all optimal closed-loop transfer functions, $\mathcal{T}_1$ can be parameterised as:

$$\mathcal{T}_1 = \{ \begin{pmatrix} s_1 \alpha(s) & 0 \\ 0 & \hat{R} + \mathcal{F}_1(\hat{Q}_a, s_1^{-1} \mathcal{BH}_\infty^{(p-1) \times (m-1)}) \end{pmatrix} X \}$$

where

$$Y := \begin{pmatrix} M & \sim \hat{N} \\ N & \sim \hat{M} \end{pmatrix}$$

and $X = W^\sim$ are square all-pass. (ii) The first two superoptimal levels of $T$ are $(\sqrt{s_1^2 + 1}, \sqrt{s_2^2 + 1})$ where $(s_1, s_2)$ are the first two super-optimal levels of $R$. Hence $\epsilon^* = \frac{1}{\sqrt{s_1^2 + 1}}$.

**Proof:** Follows from equations (8) and Theorem 3.2. \hspace{1cm} ■

B. Uniformly Destabilising perturbations

All optimal compensators contained in set $\mathcal{K}_1$ maximise the robust stability radius for the class of normalised coprime perturbations, i.e. guarantee that all $\Delta \in \mathcal{D}_\ast$ do not destabilise the system. Pick any $K \in \mathcal{K}$. As $\mathcal{D}_\ast$ is an open set, there must exist $\Delta \in \partial \mathcal{D}_\ast = \{\Delta \in \mathcal{H}_\infty : \|\Delta\|_\infty = \epsilon^*\}$ such that $(G_\Delta, K) \not\in \mathcal{S}$ (for otherwise $\epsilon^*$ would not be optimal). Such (real-rational) $\Delta$’s are constructed in [Vid85]. In the sequel we identify controllers within $\mathcal{K}_1$ that guarantee improved robust stability properties (apart from stabilising all $\Delta \in \mathcal{D}_\ast$). We first give the following definition:

**Definition:** A $\Delta \in \partial \mathcal{D}_\ast$ is called uniformly destabilising if $(G + \Delta, K) \not\in \mathcal{S}$ for every $K \in \mathcal{K}_1$. \hspace{1cm} ■

The next Lemma is ensures that (real-rational) uniformly destabilising perturbations exist on the boundary of $\mathcal{D}_\ast$. The proof of the Lemma (which is omitted) relies on a direct construction of such perturbations using the techniques of [Vid85]. The construction reveals that all frequencies are “equally critical”, in the sense that such perturbations can be constructed so that the generalised Nyquist stability criterion of the open-loop perturbed system can be violated at an arbitrary frequency (including zero and infinity).

**Lemma 3.1:** There exists $\Delta = (\Delta_N, \Delta_M) \in \partial \mathcal{D}_\ast$ such that $(\bar{M} + \Delta_M)^{-1}(\bar{N} + \Delta_N), K \not\in \mathcal{S}$ for every $K \in \mathcal{K}_1$. Furthermore, $\Delta$ can be chosen to be a stable real-rational matrix function. \hspace{1cm} ■

**Remark:** Uniformly destabilising perturbations should not be confused with “nearest unstabilisable perturbations” which refers to all $\Delta = (\Delta_N, \Delta_M) \in \mathcal{D}_\ast$ of minimum norm, say $\epsilon_{\text{max}}$, such that $(\bar{M} + \Delta_M, \bar{N} + \Delta_N)$ fails to be...
be a co-prime pair (over $C_\pm$). Such perturbations correspond to an unstable pole-zero cancellation in the perturbed plant and therefore cannot be stabilised by any controller (LTI or otherwise). In contrast, the term "uniformly destabilising" refers only to the set $K_1$. As loss of co-primeness (over $C_\pm$) implies unstabilisability we have $\epsilon^* \leq \epsilon_{max}$. In general, for the coprime-factor uncertainty case considered here (in contrast to, e.g., the unstructured additive uncertainty case), $\epsilon^* < \epsilon_{max}$ (see [MG90]).

Lemma 3.2: (i) Consider the two vectors:

$$\xi(s) = \begin{pmatrix} M(s) \\ N(s) \end{pmatrix} v(s)$$

and

$$\psi(s) = \begin{pmatrix} -N\sim(s) \\ \alpha\sim(s) \end{pmatrix} w(s)$$

where $v(s)$ and $w(s)$ are the first columns of $V(s)$ and $W(s)$, respectively, defined in Theorem 3.2. Then $\xi(s), \psi(s) \in RL_c$. $\xi\sim(s)\xi(s) = \psi\sim(s)\psi(s) = 1$ and $(\xi(s), \psi(s))$ are point-wise orthogonal, i.e. $\psi\sim(s)\xi(s) = 0$. (ii) Let

$$y_{sec}(s) = \frac{1}{\sqrt{s_1+1}} (s_1\alpha(s)\xi(s) + \psi(s)).$$

Then $y_{sec}(s) \in RL_c$ and $y_{sec}(s)_{j\omega_0}$ = 1.

Proof: (i) The result follows immediately from the fact that

$$\begin{pmatrix} M & -N\sim \\ N & \alpha\sim \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

(see equations (1) and (3)). (ii) The result follows since

$$y_{sec}(s) = \frac{1}{s_1+1} (s_1\alpha(s)\xi(s) + \psi(s))$$

is singular. Partition $X(j\omega_0)$ and $Y(j\omega_0)$ as follows:

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = X(j\omega_0)$$

and

$$Y(j\omega_0) = \begin{pmatrix} M & -N\sim \\ N & \alpha\sim \end{pmatrix} \begin{pmatrix} v & V_\perp \\ 0 & W_\perp \end{pmatrix} (j\omega_0)$$

Next, introduce a suitable permutation matrix $P$ which interchanges the second and third block columns of $\Delta(j\omega_0)$, i.e. define

$$\tilde{\Delta} = \Delta(j\omega_0)P = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24} \end{pmatrix}$$

Clearly $||\tilde{\Delta}(j\omega_0)|| = ||\Delta|| = \epsilon^*$. Then, singularity of $I - \Delta(j\omega_0)T(j\omega_0)$ is now equivalent to:

$$\det \begin{pmatrix} I_m - \tilde{\Delta} & s_1 \Psi(j\omega_0) \\ 0 & 1 \end{pmatrix} = 0$$

or

$$\det \begin{pmatrix} I_m - \tilde{\Delta} & s_1 \Psi(j\omega_0) \\ 0 & 1 \end{pmatrix} = 0$$

Assume now for contradiction that

$$1 - \tilde{\Delta}_{11}s_1 - \tilde{\Delta}_{13} \neq 0$$

Then, expanding the determinant,

$$(1 - \tilde{\Delta}_{11}s_1 - \tilde{\Delta}_{13}) \det \begin{pmatrix} I_m - \tilde{\Delta}_{22} & s_1 \Psi(j\omega_0) - \tilde{\Delta}_{24} \tilde{\Delta}_{13} \\ \tilde{\Delta}_{21}s_1 + \tilde{\Delta}_{23} & I_m - \tilde{\Delta}_{22} & \tilde{\Delta}_{13} \end{pmatrix} = 0$$

or, in LFT form:

$$\det \begin{pmatrix} I_m - \tilde{\Delta} & s_1 \\Psi(j\omega_0) - \tilde{\Delta}_{24} \tilde{\Delta}_{13} \\ \tilde{\Delta}_{21}s_1 + \tilde{\Delta}_{23} \end{pmatrix} = 0$$

and $(G_\Delta, K) \notin S$, there exists $\omega_o \in R \cup \{\infty\}$ such that

$$\det (I - \Delta(j\omega_0)T(j\omega_0)) = 0.$$
Now, from standard LFT contractive properties:

\[
\| F_u \left( \hat{\Delta}, \left( \begin{array}{c} s_1 \\ 1 \end{array} \right) \right) \| \leq \epsilon^* = \frac{1}{\sqrt{s_1^2 + 1}}
\]

Also, \(|\Psi(j\omega_o)|| < s_1\) by assumption, which implies that

\[
\left\| \left( \frac{\Psi(j\omega_o)}{1} \right) \right\| = \sqrt{\|\Psi(j\omega_o)\|^2 + 1} < \sqrt{s_1^2 + 1} = (\epsilon^*)^{-1}
\]

and so,

\[
\left\| F_u \left( \hat{\Delta}, \left( \begin{array}{c} s_1 \\ 1 \end{array} \right) \right) \right\| < 1
\]

which contradicts the singularity of the matrix in (10). Thus, contrary to the initial assumption,

\[
1 - \delta_{11}s_1 - \delta_{13} = 0 \Rightarrow \delta_{11}s_1 + \delta_{13} = 1
\]

or by direct substitution,

\[
x_1' \Delta(j\omega_o) (y_1 \alpha(j\omega_o)s_1 + y_2) = 1
\]

\[
\Rightarrow x'(j\omega_o) \Delta(j\omega_o)y_{sc}(j\omega_o) = 1
\]

\[
= \frac{1}{\sqrt{s_1^2 + 1}} = \epsilon^*
\]

where \(x'(s)\) denotes the first row of \(X(s)\) and \(y_{sc}(s)\) is defined in Lemma 3.2. Thus \(|x'(s)\Delta(s)y_{sc}(s)||_{\infty} \geq \epsilon^*\). However, since \(x'^* = y_{sc}'y_{sc} = 1\) and \(\|\Delta\|_{\infty} = \epsilon^*\) we conclude that \(|x'\Delta y_{sc}|_{\infty} = \epsilon^*\).

**Remark:** We can interpret the condition \(x'(j\omega_o)\Delta(j\omega_o)y_{sc}(j\omega_o) = \epsilon^*\) as follows: Define an inner product over \(\mathbb{C}^{p \times m}\) (the space of \(p \times m\) complex matrices) as:

\[
\langle A, B \rangle = \text{trace}(B^*A)
\]

whenever \(A, B \in \mathbb{C}^{p \times m}\). Then we can write:

\[
x'(j\omega_o)\Delta(j\omega_o)y_{sc}(j\omega_o) = \text{trace}(y_{sc}(j\omega_o)x'(j\omega_o)\Delta(j\omega_o))
\]

\[
= \langle \Delta(j\omega_o), E_o \rangle = \epsilon^*
\]

where \(E_o := x(-j\omega_o)y_{sc}'(-j\omega_o)\), which means that \(\Delta\) has a projection of \(\epsilon^*\) in the direction defined by \(E_o\).

**C. Extended robust stability radius**

Lemma 3.3 shows that all uniformly destabilising perturbations \(\Delta\) are constrained to have a projection equal to \(\epsilon^*\) along the fixed direction \(x(-j\omega_o)y_{sc}'(-j\omega_o)\) at some frequency \(\omega_o\). This means that it is impossible to extend the robust stability radius along this direction, using a subset of all maximally robust controllers \(K_1\) (assume that we still want to stabilise all \(\Delta \in \mathcal{D}_{\epsilon^*}\)). Moreover, all frequencies are equally critical, in the sense that we can construct uniformly destabilising perturbations such that the generalised Nyquist criterion is violated at an arbitrary frequency. Thus, we can only hope to extend the robust stability radius (beyond \(\epsilon^*\)) at directions other than \(\{\omega: x(-j\omega)y_{sc}'(-j\omega), \omega \in \mathbb{R} \cup \{\infty\}\}\).

To motivate the formulation of an optimisation problem which allows us to extend the robust stability radius in all directions (other than the “most critical” direction), consider the following “distance to singularity” problem:

Let \(A\) be a \(n \times n\) complex non-singular matrix with singular value decomposition \(A = USV^\star = \sum_{i=1}^{n} \sigma_i u_i v_i^\star\) with \(\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\), \(\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{n-1} > \sigma_n > 0\). What is the minimum norm perturbation \(|E|\) such that \(A - E\) is singular? It is well known that the unique solution is given by the rank 1 matrix \(E_o = \sigma_n u_n v_n^\star\) so that \(|E_o| = \sigma_n\). Thus in this case \(u_n^\star v_n^\star = \sigma_n^\star\). Suppose now that we constrain the magnitude of the projection of allowable perturbations in this direction, i.e. impose the restriction that

\[
|\langle u_n^\star v_n^\star, E \rangle| \leq \phi
\]

for some non-negative constant \(\phi \leq \sigma_n\). Since now the new minimum-norm singularising perturbation cannot have a projection of magnitude \(\sigma_n\) in the most-critical direction, we expect the constrained optimal distance to singularity \(\gamma(\phi)\) to be larger than \(\sigma_n\); further, the tighter the constraint (\(\phi\) decreases), the more \(\gamma(\phi)\) should deviate from \(\sigma_n\). The full solution to the problem is provided by the following Lemma.

**Lemma 3.4:** Let \(A\) be a square non-singular complex matrix which has a singular value decomposition \(A = U\Sigma V^\star = \sum_{i=1}^{n} \sigma_i u_i v_i^\star\), where \(\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\), \(\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{n-2} \geq \sigma_{n-1} > \sigma_n > 0\). Then all \(E\) which minimise \(\gamma(\phi) = \min \{|E|: \text{det}(A - E) = 0, |\langle u_n^\star v_n^\star, E \rangle| \leq \phi \leq \sigma_n\}\)

are given by:

\[
E = U \begin{bmatrix} \phi & 0 & 0 \\ \nu^* & -\phi & 0 \\ 0 & 0 & P_s \end{bmatrix} V^\star
\]

where \(P_s\) is arbitrary except for the constraint

\[
\|P_s\| \leq \sqrt{\sigma_n\sigma_{n-1} + \phi(\sigma_n - \sigma_{n-1})}
\]

and \(\nu\) is given by

\[
\nu = \sqrt{(\phi + \sigma_{n-1})(\sigma_n - \phi)} e^{j\theta}, \quad \theta \in [0, 2\pi].
\]

The minimum value of \(\gamma(\phi)\) is given by the righthand side of (11).

**Proof:** See [LCLS’84]. For a number of generalisations to the problem see [HMG06].

**Remark:** In the above formulation of the problem \(\sigma_n\) and \(\sigma_{n-1}\) are fixed and so the constrained distance to singularity \(\gamma(\phi)\) is a function only of \(\phi\). Suppose that somehow we could influence the level of \(\sigma_{n-1}\), assuming that \(\sigma_n\) and \(\phi\) are fixed. Then, in order to maximise \(\gamma(\phi)\), we would have to maximise \(\sigma_{n-1}\), i.e. make the gap \(\sigma_{n-1} - \sigma_n\) as large as possible, an observation which motivates super-optimisation used later in the section.

Motivated by the above result we proceed as follows: Suppose we impose a structure on the permissible uncertainty set, by defining the set:

\[
E(\delta, \mu) = \{ \Delta \in \mathcal{D}_\mu : \|x'\Delta y_{sc}\|_{\infty} \leq (1 - \delta)\epsilon^* \}
\]

where

\[
\mathcal{D}_\mu = \{ \Delta \in \mathcal{H}_\infty : \|\Delta\|_{\infty} < \mu \}
\]

Then we formulate the following optimisation problem:
Constrained maximum robust stabilisation (CMRS): For a fixed $\delta$, $0 \leq \delta \leq 1$, find all $K$ that solve:

$$\sup\{\theta : (G\Delta, K) \in S \}$$

and the corresponding value of the supremum $\mu = \mu^*(\delta)$.

Remark: (i) Note that since we still require that all $\Delta \in D_\epsilon$ are stabilised, the set of optimal controllers which solve CMRS must be a subset of $K_1$. (ii) When $\delta = 0$ the constraint $\|x'\Delta y_{sc}\|_\infty \leq (1 - \delta)e^*$ is redundant (i.e. no structure is imposed) and thus $E(0, \mu) = D_\mu$; hence in this case the solution to the CMRS problem is trivial and is given by $K_{opt} = K_1$ and $\mu^*(0) = e^*$.

The solution of the CMRS problem is summarised in the following Theorem which is the main result of the paper. The Theorem is stated without a proof due to lack of space. Note that $(s_1, s_2)$ denote the first two super-optimal levels of $R$ and we assume that $s_1 > s_2$. Further, $K_1$ denotes the set of all optimal (maximally-robust) controllers and $K_2$ the set of all super-optimal controllers with respect to the first two levels, so that $K_2 \subseteq K_1$.

Theorem 3.4: In previously defined notation the following statements hold:

1) For each $\delta \in [0, 1],$

$$\mu^*(\delta) = \frac{1}{\sqrt{s_1^2 + 1}} \left( \frac{\delta}{\sqrt{s_2^2 + 1}} + \frac{1 - \delta}{\sqrt{s_2^2 + 1}} \right) \geq e^*$$

with equality only in the case $\delta = 0$. Here $\sqrt{s_1^2 + 1}$ and $\sqrt{s_2^2 + 1}$ are the first two (distinct) super-optimal levels of $T$ with $\sqrt{s_1^2 + 1} = (e^*)^{-1}$.

2) For each $0 < \delta \leq 1$ the following two statements are equivalent:

(a) $$(\tilde{M} + \Delta M)^{-1}(\tilde{N} + \Delta N), K) \in S$$

for every $\Delta \in D_\epsilon$, $\epsilon \in \mathcal{E}(\delta, \mu^*(\delta))$,

(b) $K \in K_2$.

3) (a) $E(0, \mu^*(0)) = D_\epsilon$,

(b) for each $K \in K_2$,

$$(\tilde{M} + \Delta M)^{-1}(\tilde{N} + \Delta N), K) \in S$$

for every $\Delta \in \bigcup_{\delta \in [0, 1]} \mathcal{E}(\delta, \mu^*(\delta))$.

Remark: (i) As expected the constrained robust stability radius $\mu^*(\delta)$ is a strictly increasing function of $\delta$ with $\mu^*(0) = e^*$. Moreover, for a fixed $\delta \neq 0$ and $s_1$, $\mu^*(\delta)$ is a decreasing function of $s_2$. Thus the structured robust stability radius $\mu^*(\delta)$ increases with an increasing gap between the first two superoptimal levels. (ii) For each $\delta \neq 0$ the set of optimal controllers is the same, namely $K_2$. Thus each super-optimal controller guarantees the stability of all perturbations in the union of the sets $\bigcup_{\delta \in [0, 1]} \mathcal{E}(\delta, \mu^*(\delta))$ which contains the ball of radius $e^*$ as a subset.

Remark: When the model uncertainty set is unstructured, Theorem 3.4 shows that using a superoptimal controller (with respect to the first two levels) guarantees robust stabilisation for a larger class of uncertainties $\bigcup_{\delta \in [0, 1]} \mathcal{E}(\delta, \mu^*(\delta))$ compared to the class guaranteed to be stabilised by using an arbitrary optimal controller ($D_\epsilon$). In the case when the uncertainty set is structured, we can give the following interpretation of Theorem 3.4: Consider a normalised structured uncertainty set $B_\Delta_S = \{\Delta \in S : \|\Delta\| \leq 1\} \subseteq B_\mathcal{H}_\infty$, where $S$ is an arbitrary structure. Suppose that we can determine the maximum value of $\delta$ in the interval $[0, 1]$, say $\delta^*$, such that $\|x'\Delta y\|_\infty \leq 1 - \delta^*$ for all $\Delta \in B_\Delta_S$. Since there exists a controller $K \in K_2$ which stabilises all $\Delta$ such that (i) $\|\Delta\|_\infty < \mu^*(\delta^*)$ and (ii) $\|x'\Delta y_{sc}\|_\infty \leq e^*(1 - \delta^*)$, then $\mu^*(\delta^*)$ is a lower bound of the robust stability radius relative to structure $S$, which is tighter than the “unstructured” bound, i.e. $\mu^*(\delta^*) > e^*$, provided $\delta^* \neq 0$. This approach can be used to breach the convex upper bound of the structured singular value under complex block-structured uncertainties (see [JHMG06] for details).

IV. Conclusions

The paper has considered a maximum robust stabilisation problem for multivariable systems under normalised co-prime factor perturbations. It was shown that the problem can be reduced to a Nehari extension problem, the solution of which gives the maximum robust stability radius and a complete parametrisation of all optimal (maximally robust) controllers. By analysing the properties of the optimal solution, a “worst-case” direction was identified, along which all boundary uniformly-destabilising perturbations were shown to lie. By imposing a parametric constraint on the projection of admissible perturbations along this direction (uniformly in frequency), it was shown that it is possible to extend the robust stability radius in every other direction, using a subset of all optimal (maximally-robust) controllers, by solving a
two-level super-optimal Nehari extension problem. The paper extends the approach of [GHJ00] for additive uncertainty models to the co-prime uncertainty case. Problems of this type have proved to be important both in terms of analysis and robust control design ($\nu$-gap metric [Vin00] and loop shaping design methodology [MG90]). In future work we will attempt to establish links with these areas. Furthermore, we will attempt to relax the assumption made in this work related to the multiplicity of the largest Hankel singular value. Problems involving multiplicities have recently been shown to be significant in the area of structured uncertainty [JHMG06].

REFERENCES


