TUBE-BASED ROBUST NONLINEAR MODEL PREDICTIVE CONTROL [1]

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Abstract: A successful method for model predictive control of constrained linear systems uses a local linear control law that, in the presence of disturbances, maintains the disturbed trajectory of the controlled system in a tube whose centre is the nominal trajectory (generated ignoring the disturbance) and whose 'cross-section' is a robust, positively invariant set; robust exponential stability of an invariant set centred on the origin may be established. The purpose of this paper is to show how this successful procedure may be extended to provide robust model predictive control of constrained nonlinear systems. An ancillary problem is proposed, the solution of which provides a local nonlinear control law. The disturbed trajectories lie in a tube and this provides the means for constructing a tube-based robust nonlinear model predictive controller.

Keywords: Robust Model Predictive Control, Nonlinear Model Predictive Control, Discrete-Time Systems, Tubes, Uncertain Systems.

1. INTRODUCTION

Uncertainty demands that the optimal control problem employed in model predictive control should have a (feedback) control policy rather than an (open-loop) control sequence as a decision variable (Mayne *et al.*, 2000). To reduce the unmanageable complexity of the resultant optimal control problem it is possible to employ, for constrained linear systems, a tube whose centre is the trajectory of a nominal system and whose 'cross section' is a robust positively invariant set (Mayne *et al.*, 2005). Robust exponential stability of a robust positively invariant set is achieved with a controller that has similar complexity to the optimal control problem conventionally used in model predictive control. This paper show how this procedure may be extended to provide robust model predictive control of constrained nonlinear systems.

The system to be controlled is assumed to be described by a nonlinear difference equation with an additive bounded disturbance:

$$x^+ = f(x,u) + w \tag{1.1}$$

where $f(\cdot)$ is three times continuously differentiable; the system is required to satisfy the state and control constraints

$$x \in \mathbb{X}, \quad u \in \mathbb{U} \tag{1.2}$$

where $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are C-sets (compact sets containing the origin in their interior). The solution of (1.1) at time i if the initial state (at time 0) is x, and the control is generated by policy π is $\phi(i; x, \pi, \mathbf{w})$ where $\mathbf{w}$ denotes the disturbance sequence $\{w(0), w(1), \ldots\}$. The nominal system is described by

$$z^+ = f(z,v) \tag{1.3}$$

and its solution at time i if its initial state is z is denoted by $\bar{\phi}(i; z, \mathbf{v})$ where $\mathbf{v} \triangleq \{v(0), v(1), \ldots\}$ is the nominal control sequence. The deviation between the actual and nominal state is $e \triangleq x - z$ and satisfies

$$e^+ = f(x,u) - f(z,v) + w. \tag{1.4}$$

Because $f(\cdot)$ is nonlinear we have to proceed more indirectly than in the linear case. We use, as before (Mayne *et al.*, 2005), a nominal controller

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with tighter constraints than in the original problem to generate a central path and replace the pre-derived stabilizing controller $u = v + K(x - z)$ (employed in the linear case to keep all solutions of the uncertain system in an invariant set centered on the nominal trajectory) by an ancillary controller that uses a modified model predictive controller to keep solutions close to the central path. We make use of the fact that, as in the linear case, it is not necessary to know the invariant set; we merely need an outer bound for it.

The approach presented in this paper can be compared with a number of existing robust nonlinear MPC schemes. To overcome the computational complexities of min-max approaches to robust nonlinear MPC (Magni *et al.*, 2003), some methods guarantee robustness by tightening the constraints of the nominal problem off-line by a sufficient amount, either by exploiting the continuity (Grimm *et al.*, 2003) or Lipschitz continuity (Marruedo *et al.*, 2002) of the dynamics. Other papers tighten the constraints on-line by computing bounds on the set of trajectories reachable from the current state, either by linearizing around trajectories (Lee *et al.*, 2002) or via the computation of approximations to the reachable set (Limon *et al.*, 2005; Bravo *et al.*, 2006). The constraints can also be tightened on-line by performing a suitable sensitivity analysis of the finite horizon optimal control problem (Ma and Braatz, 2001; Nagy and Braatz, 2003; Diehl *et al.*, 2006). In this paper, we also propose to tighten the constraints of the nominal problem — this can be done off-line (Grimm *et al.*, 2003; Grimm *et al.*, 2003) or on-line (Ma and Braatz, 2001; Nagy and Braatz, 2003; Diehl *et al.*, 2006), but the exact details of how to achieve this is not particularly important to the method proposed below. The main difference is that the new method involves formulating and solving an ancillary MPC problem, which serves to contain the trajectories of the actual system in a tube around the nominal trajectory. This provides a stronger guarantee of robust stability than is possible with existing constraint-tightening methods, which only guarantee stability of the origin or a bounded set containing the origin (Marruedo *et al.*, 2002; Grimm *et al.*, 2003). The new method is also computationally more attractive than bounding the set of reachable states on-line (Lee *et al.*, 2002; Limon *et al.*, 2005; Bravo *et al.*, 2006) or performing a min-max optimization over feedback policies (Magni *et al.*, 2003). (Rakovic *et al.*, 2006) show how robust control may be obtained for nonlinear systems with special structure.

2. SIMPLE ROBUST MODEL PREDICTIVE CONTROLLER

We describe in this section a robust model predictive controller for nonlinear systems that has two components: a *nominal* controller that generates a central path and an *ancillary* controller that endeavours to steer the trajectories of the uncertain system to the central path.

2.1 Nominal controller

The cost function $\bar{V}_N(\cdot)$ for the nominal optimal control problem is defined by

$$\bar{V}_N(z, \mathbf{v}) \triangleq V_f(z(N)) + \sum_{k=0}^{N-1} \ell(z(k), v(k)) \quad (2.1)$$

where $z(k) = \bar{\phi}(k; z, \mathbf{v})$, z is the initial state and $\mathbf{v}$ the nominal control sequence and $|\cdot|$ denotes the usual Euclidean norm or induced norm. The function $\ell(\cdot)$ is defined by

$$\ell(x, u) \triangleq (1/2)|x|_Q^2 + (1/2)|u|_R^2 \quad (2.2)$$

where Q an R are positive definite, $|x|_Q \triangleq (x^T Q x)^{1/2}$, etc. We impose the following state and control constraints on the nominal system:

$$z \in \mathbb{Z}, \quad v \in \mathbb{V}, \quad (2.3)$$

where $\mathbb{Z} \subset \mathbb{X}$ and $\mathbb{V} \subset \mathbb{U}$; the terminal cost function $V_f(\cdot)$ together with the terminal constraint set $Z_f \subseteq \mathbb{X}$ are chosen as described in (Mayne *et al.*, 2000) (to satisfy the 'stability axioms'). We do not wish to be too prescriptive in the choice of $\mathbb{Z}$ and $\mathbb{V}$ since, unlike in the linear case, we do not have an invariant set that bounds $e = x - z$. For example, if $\mathbb{X}$ and $\mathbb{U}$ are convex we can choose $\mathbb{Z} = \alpha\mathbb{X}$ and $\mathbb{V} = \beta\mathbb{U}$; $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ are scalar tuning parameters whose choice we discuss below. The state and control constraints, and the terminal constraint $z(N) \in Z_f$ impose a constraint $\mathbf{v} \in \mathcal{V}_N(z)$ on the nominal control sequence where

$$\begin{aligned} \mathcal{V}_N(z) \triangleq \{\mathbf{v} \mid\ & v(k) \in \mathbb{V} \text{ and } \bar{\phi}(k; z, \mathbf{v}) \in \mathbb{Z} \\ & \forall k \in \{0, 1, \ldots, N-1\}, \bar{\phi}(N; z, \mathbf{v}) \in Z_f\}. \end{aligned} \quad (2.4)$$

For each z, the set $\mathcal{V}_N(z)$ is compact (bounded because of the assumptions on $\mathbb{V}$ and closed (because of the continuity of $\bar{\phi}(\cdot)$). The nominal optimal control problem $\bar{\mathbb{P}}_N(z)$ is defined by

$$\bar{\mathbb{P}}_N(z): \quad \bar{V}_N^0(z) = \min_{\mathbf{v}} \{\bar{V}_N(z, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_N(z)\}. \quad (2.5)$$

A solution exists because $\bar{V}_N(\cdot)$ is continuous and $\mathcal{V}_N(z)$ is compact. Let $Z_N \triangleq \{z \mid \mathcal{V}_N(z) \neq \emptyset\}$ denote the domain of $\bar{V}_N^0(z)$, the set of feasible states for $\bar{\mathbb{P}}_N(z)$; by virtue of our assumptions, the set Z_N is bounded. The solution of $\bar{\mathbb{P}}_N(z)$ is the minimizing control sequence

$$\mathbf{v}^0(z) = \{v^0(0; z), v^0(1; z), \ldots, v^0(N-1; z)\}$$

which we assume is unique, and the associated optimal state sequence is

$$\mathbf{z}^0(z) = \{z, z^0(1;z), \ldots, z^0(N;z)\}.$$

The first element $v^0(0;z)$ of $\mathbf{v}^0(z)$ is the control that is applied in model predictive control. The implicit model predictive control law is, therefore, $\bar{\kappa}_N(\cdot)$ defined by

$$\bar{\kappa}_N(z) \triangleq v^0(0;z). \tag{2.6}$$

The nominal system under model predictive control satisfies

$$z^+ = f(z, \bar{\kappa}_N(z)). \tag{2.7}$$

It follows from the definition of $\bar{V}_N^0(\cdot)$ that there exists a $c_1 > 0$ such that

$$\bar{V}_N^0(z) \geq c_1|z|^2 + c_1|\mathbf{v}^0(z)|^2 \tag{2.8}$$

for all $z \in Z_N$. If the terminal cost function $V_f(\cdot)$ and terminal constraint set Z_f are chosen to satisfy certain stability assumptions, which we assume to be the case, and Z_N is bounded, there exists a $\bar{c}_2 > c_1$ such that

$$\bar{V}_N^0(z) \leq \bar{c}_2|z|^2, \text{ and} \tag{2.9}$$

$$\Delta\bar{V}_N^0(z, \bar{\kappa}_N(z)) \leq -c_1|z|^2 \tag{2.10}$$

for all $z \in Z_N$ where

$$\Delta\bar{V}_N^0(z,v) \triangleq \bar{V}_N^0(f(z,v)) - \bar{V}_N^0(z).$$

It follows that the origin is exponentially stable for the system $z^+ = f(z, \bar{\kappa}_N(z))$ and Z_N is a region of attraction. For future use, we note that (2.8) and (2.9) imply the existence of a $\bar{c} > 0$ such that

$$|\mathbf{v}^0(z)| \leq \bar{c}|z|, \quad \forall z \in Z_N. \tag{2.11}$$

2.2 Ancillary controller

The purpose of the ancillary controller is to maintain the state of the uncertain system $x^+ = f(x,u) + w$ close to the trajectory of the nominal system $z^+ = f(z, \bar{\kappa}_N(z))$; the ancillary controller replaces the pre-computed controller $u = v + K(x-z)$ employed in the linear case. To obtain the ancillary controller, we determine a model predictive controller that minimizes the cost of the deviation between the trajectories of the two systems $x^+ = f(x,u)$ and $z^+ = f(z, \bar{\kappa}_N(z))$, i.e. we omit the disturbance w from the uncertain system and rely on the stabilizing property of the resultant controller to constrain the deviation between the trajectories of the systems $x^+ = f(x,u) + w$ and $z^+ = f(z, \bar{\kappa}_N(z))$.

The ancillary controller is, therefore, based on the composite system:

$$x^+ = f(x,u), \tag{2.12}$$

$$z^+ = f(z, \bar{\kappa}_N(z)). \tag{2.13}$$

The cost $V_N(x,z,\mathbf{u})$ that measures the distance between the trajectories of these two systems is defined by:

$$V_N(x,z,\mathbf{u}) \triangleq \sum_{i=0}^{N-1} \ell(x(i) - z^*(i;z), u(i) - v^*(i;z)) \tag{2.14}$$

where $x(i) \triangleq \bar{\phi}(i;x,\mathbf{u})$ is the solution of (2.12) at time i if the initial state is x and the control input sequence is $\mathbf{u}$; $z^*(i;z)$ is the solution of (2.13) at time i if the initial state is z and $v^*(i;z) \triangleq \bar{\kappa}_N(z^*(i;z))$. For later use, we define the sequences $\mathbf{v}^*(z) \triangleq \{v^*(0;z), v^*(1;z), \ldots v^*(N-1;z)\}$ and $\mathbf{z}^*(z) \triangleq \{z^*(0;z), z^*(1;z), \ldots z^*(N;z)\}$ where $z^*(0;z) = z$ and $v^*(0;z) = \bar{\kappa}_N(z)$. For simplicity, we have chosen the cost function $\ell(\cdot)$ in (2.14) to be the same function as in the definition (2.1) of the cost for the nominal controller; this choice is not necessary. The ancillary control problem is the minimization of $V_N(x,z,\mathbf{u})$ with respect to $\mathbf{u}$ subject to the control constraints and terminal equality constraint $x(N) = z^*(N;z)$; this terminal constraint is chosen for simplicity.

Hence, the ancillary control problem $\mathbb{P}_N(x,z)$ is defined by:

$$V_N^0(x,z) = \min_{\mathbf{u}}\{V_N(x,z,\mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x,z)\}$$

$$\mathcal{U}_N(x,z) \triangleq \{\mathbf{u} \mid \bar{\phi}(N;x,\mathbf{u}) = z^*(N;z), \; u(i) \in \mathbb{U} \; \forall i \in \{0,1,\ldots,N-1\}\} \tag{2.15}$$

where $\mathcal{U}_N(x,z)$ is the constraint set. For each (x,z), the set $\mathcal{U}_N(x,z)$ is compact. There is no terminal cost and the terminal constraint set (set of permissible terminal states) is a single state:

$$X_f(z) = \{\bar{\phi}(N;z,\mathbf{v}^*(z))\} = \{z^*(N;z)\}.$$

For each $z \in Z_N$, the domain of the value function $V_N^0(\cdot,z)$ (and of the minimizer) is the set $X_N(z)$ defined by

$$X_N(z) \triangleq \{x \mid \mathcal{U}_N(x,z) \neq \emptyset\}.$$

For each $z \in Z_N$, the set $X_N(z)$ is bounded. For future reference, let the set $\Pi_N \subset \mathbb{R}^n \times \mathbb{R}^n$ be defined by

$$\Pi_N \triangleq \{(x,z) \mid z \in Z_N, \; x \in X_N(z)\}.$$

The set Π_N is bounded. For any $(x,z) \in \Pi_N$, the minimizing control sequence is $\mathbf{u}^0(x,z) = \{u^0(0;x,z), u^0(1;x,z), \ldots, u^0(N-1;x,z)\}$ and the control applied to the system is $u^0(0;x,z)$, the first element in this sequence. The corresponding optimal state sequence is $\mathbf{x}^0(x,z) = \{x, x^0(1;x,z), \ldots, x^0(N;x,z)\}$. The implicit ancillary control law is, therefore, $\kappa_N(\cdot)$ defined by:

$$\kappa_N(x,z) \triangleq u^0(0;x,z). \tag{2.16}$$

If $x = z$, then, as is easily verified,

$$u^0(i;x,z) = v^*(i;z), \; i = 0,1,\ldots,N-1 \tag{2.17}$$

so that the control and state trajectories of the two systems (2.12), (2.13) are identical and

$$\kappa_N(z, z) = \bar{\kappa}_N(z). \tag{2.18}$$

3. CONTROLLER IMPLEMENTATION

Suppose $\mathbb{Z}$ and $\mathbb{V}$ have been chosen so that $\mathbb{Z}$ lies in the interior of $\mathbb{X}$ and $\mathbb{V}$ lies in the interior of $\mathbb{U}$. The controller algorithm is:

Robust control algorithm

Data: Current state (x, z) of the system.
Step 1: Determine the sequences $\mathbf{z}^*(z)$ and $\mathbf{v}^*(z)$ where $z^*(0; z) = z$ and $v^*(0; z) = \bar{\kappa}_N(z)$.
Step 2: Solve the ancilliary problem $\mathbb{P}_N(x, z)$ to obtain the current control $u = \kappa_N(x, z)$.
Step 3: Apply the control $u = \kappa_N(x, z)$ to the system being controlled and measure the successor state $x^+ = f(x, \kappa_N(x, z)) + w$.
Step 4: Set $z^+ = z^*(1; z) = f(z, \bar{\kappa}_N(z))$.
Step 5: Set $(x, z) = (x^+, z^+)$ (the new current state) and go to Step 1.

In Step 1, after initialization, the sequences $\mathbf{z}^*(z)$ and $\mathbf{v}^*(z)$ may be determined from the previous sequences by solving $\bar{\mathbb{P}}_N$ once. In the linear case $u = v + K(x - z)$, $\mathbb{Z} = \mathbb{X} \ominus S$ and $\mathbb{V} = \mathbb{U} \ominus KS$ where S is a pre-computed invariant set. In the nonlinear case, it is possible to formulate a global optimization problem to choose $\mathbb{Z} = \alpha\mathbb{X}$ and $\mathbb{V} = \beta\mathbb{U}$.

4. PROPERTIES OF THE CONTROLLED SYSTEM

It follows from the definition of $V_N(\cdot)$ that there exists a $c_1 > 0$ such that

$$V_N^0(x, z) \geq c_1|x - z|^2 + c_1|\mathbf{u}^0(x, z) - \mathbf{v}^*(z)|^2 \tag{4.1}$$

for all $(x, z) \in \Pi_N$; in (4.1), $\mathbf{u}^0(x, z)$ is the minimizing control sequence for $\mathbb{P}_N(x, z)$. Let $\theta(\cdot)$ be defined by:

$$\theta(x, z, \mathbf{u}) \triangleq \bar{\phi}(N; x, \mathbf{u}) - z^*(N; z)) \tag{4.2}$$

so that $\theta(x, z, \mathbf{u}) = 0$ implies $\mathbf{u} \in \mathcal{U}_N(x, z)$; $f(\cdot)$ twice continuously differentiable implies the function $(x, \mathbf{u}) \mapsto \theta(x, z, \mathbf{u})$ is twice continuously differentiable for each $z \in Z_N$. We require the following controllablity assumption:

Assumption 1. (i) There exists a $c_2 > 0$ such that

$$V_N^0(x, z) \leq c_2|x - z|^2 \ \forall (x, z) \in \Pi_N. \tag{4.3}$$

(ii) The function $f(\cdot)$ is twice continuously differentiable and, for all $(x, z) \in \Pi_N$,

$$(\partial/\partial\mathbf{u})\theta(x, z, \mathbf{u}^0(x, z)) = (\partial/\partial u)\bar{\phi}(N; x, \mathbf{u})$$

has full rank n where $(\partial/\partial\mathbf{u})\theta$ denotes the matrix whose ijth element is $\partial\theta_i/\partial\mathbf{u}_j$ with $\mathbf{u}_j$ denoting the jth element of $\mathbf{u}$ regarded as a vector.

Assumption (i) is a 'global' and (ii) a 'local' controllability assumption. We note for future use that (4.1) and (4.3) imply

$$|\mathbf{u}^0(x, z) - \mathbf{v}^*(z)| \leq c|x - z| \tag{4.4}$$

for all $(x, z) \in \Pi_N$ where $c = ((c_2/c_1) - 1)^{(1/2)}$.

Some properties of the value function are given in the next result:

Proposition 1. The value function $V_N^0(\cdot)$ satisfies:

$$V_N^0(x, z) \geq c_1|x - z|^2, \tag{4.5}$$
$$V_N^0(x, z) \leq c_2|x - z|^2, \tag{4.6}$$
$$\Delta V_N^0(x, z, \kappa_N(x, z)) \leq -c_1|x - z|^2, \text{ and} \tag{4.7}$$
$$V_N^0(f(x, \kappa_N(x, z)), f(z, \bar{\kappa}_N(z))) \leq \gamma V_N^0(x, z) \tag{4.8}$$

for all $(x, z) \in \Pi_N$ where $\gamma \triangleq 1 - (c_1/c_2) \in (0, 1)$ and $\Delta V_N^0(x, z, \kappa_N(x, z)) \triangleq V_N^0(f(x, \kappa_N(x, z)),$ $f(z, \bar{\kappa}_N(z))) - V_N^0(x, z)$.

The value function also has some continuity and differentiable properties that be summarised as follows:

Proposition 2. For all $z \in Z_N$ there exists an $\varepsilon(z) > 0$ such that (i) the function $x \mapsto V_N^0(x, z)$ is continuously differentiable and, therefore, Lipschitz continuous in $\{z\} \oplus \varepsilon(z)\mathcal{B}$, where $\mathcal{B}$ is the unit ball in $\mathbb{R}^n$ and, (ii) the function $V_N^0(\cdot)$ is continuous at $(x, 0)$ for all $x \in \varepsilon(0)\mathcal{B}$.

For all $z \in Z_N$, all $d > 0$ let the set $S_d(z)$ be defined by:

$$S_d(z) \triangleq \{x \mid V_N^0(x, z) \leq d\}. \tag{4.9}$$

The set $S_d(z)$ is a level set of $x \mapsto V_N^0(x, z)$ and has the following properties:

Proposition 3. For all $z \in Z_N$, (i) there exists a $d(z) > 0$ such that

$$S(z) \triangleq S_{d(z)}(z) \subseteq \{z\} \oplus \varepsilon(z)\mathcal{B}.$$

where $\varepsilon(z)$ is defined in Proposition 2, (ii) for all $d > 0$, all $x \in S_d(z)$,

$$V_N^0(f(x, \kappa_N(x, z)) + w, f(z, \bar{\kappa}_N(z))) \leq d$$

for all w satisfying $|w| \leq (1 - \gamma)d/k(z)$ where $k(z)$ is the Lipschitz constant for $x \mapsto V_N^0(x, z)$ in $S_d(z)$ (cf Proposition 2).

5. A TUBE FOR THE NONLINEAR SYSTEM

We now construct a tube in which the state of the controlled system lies for all realizations of the disturbance sequence. The composite system (system and controller) satisfies

$$x^+ = f(x, \kappa_N(x, z)) + w \tag{5.1}$$
$$z^+ = f(z, \bar{\kappa}_N(z)). \tag{5.2}$$

To define the tube, we require, because of non-linearity, an extension of the normal definition of robust positive invariance.

Definition 1. Let $d > 0$ be given. The point-to-set map $z \mapsto S_d(z)$ is said to be x-robust positively invariant for the composite system (5.1)–(5.2) satisfying $w \in \mathbb{W}$ if, for all $z \in Z_N$, $x \in S_d(z)$ implies $x^+ = f(x, \kappa_N(x, z)) + w \in S_d(f(z, \bar{\kappa}_N(z))$ for all $w \in \mathbb{W}$.

The significance of this set lies in the fact that, if $x(0) \in S_d(z(0))$, then, for all $i \in \mathbb{Z}_{\geq 0}$, $x(i) \in S_d(z(i))$ for all admissible disturbance sequences; here $x(i) = \phi(i; x(0), \kappa_N(\cdot), \mathbf{w})$ is the solution at time i of $x^+ = f(x, \kappa_N(x, z), w)$ for a given disturbance sequence $\mathbf{w} \in \mathbb{W}^i$ and $z(i) = \bar{\phi}(i; z(0), \bar{\kappa}_N(\cdot))$ is the solution at time i of the nominal system $z^+ = f(z, \bar{\kappa}_N(z))$. To proceed, we require

Assumption 2. There exists a $d > 0$ such that: (i) for all $z \in Z_N$, $S_d(z) \subset S(z)$ (which is true if $d \leq d(z)$), and, (ii) $\mathbb{W}$ is such that $|w| \leq (1 - \gamma)d/k(z)$ for all $w \in \mathbb{W}$, all $z \in Z_N$.

We could, alternatively, replace the constant set $\mathbb{W}$ by the set-valued function $\mathbb{W}(z) \triangleq \{w \mid |w| \leq (1 - \gamma)d/k(z)\}$. In the sequel d is assumed to satisfy Assumption 2; under this assumption the following result is a direct consequence of Proposition 3:

Proposition 4. The point-to-set map $z \mapsto S_d(z)$ is $x-$robust positively invariant for the composite system $(x^+, z^+) = (f(x, \kappa_N(x, z)) + w, f(z, \bar{\kappa}_N(z))$.

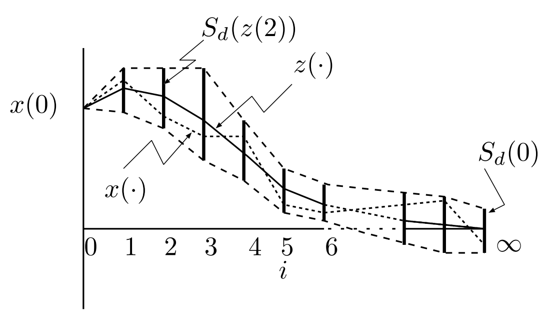

Fig. 1. Tube for a nonlinear system

The main result of this section is

Theorem 1. Suppose that the initial state (x, z) of the composite system $(x^+, z^+) = (f(x, \kappa_N(x, z)) + w, f(z, \bar{\kappa}_N(z)))$ satisfies $x \in S_d(z)$ and $z \in Z_N$. Then: (i) every solution of the uncertain system $x^+ = f(x, \kappa_N(x, z)) + w$, $w \in \mathbb{W}$, lies in the tube $\{S_d(z), S_d(z^0(1; z)), S_d(z^0(2; z)), \ldots\}$ for all $\mathbf{w} = \{w(0), w(1), \ldots w(i-1)\} \in \mathbb{W}^i$, all $i \in \mathbb{Z}_{\geq 0}$, and, (ii) every accumulation (cluster) point of every solution of the uncertain system $x^+ = f(x, \kappa_N(x, z)) + w$ lies in the positively invariant set $S_d(0)$.

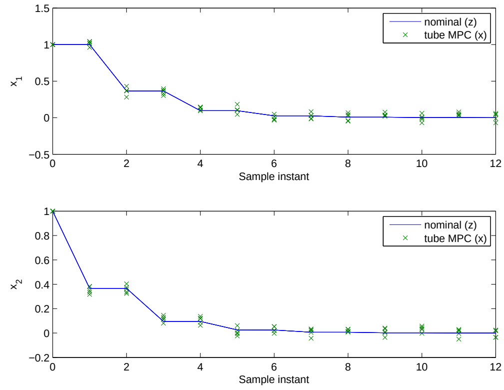

Fig. 2. Time plot of the states of the example system starting from initial condition $x = (1, 1)$. The solid line shows the trajectory of the nominal system (5.2) and the crosses show the state evolution for the actual system (5.1) for five random disturbance sequences taken from the set $\mathbb{W} := \{w \mid |w|_\infty \leq 0.1\}$.

6. EXAMPLE

Consider the following simple example with one input and two states:

$$x_1^+ = x_2 + w_1, \quad x_2^+ = \sin(x_1) + u + w_2,$$

where we have no state constraints, but only the input constraints $\mathbb{U} := \{u \mid |u| \leq 0.5\}$. For simplicity, we choose the terminal constraint for the nominal problem to be the origin $Z_f := \{0\}$ and the terminal cost to be zero $V_f(z) := 0$. The input constraints for the nominal problem are set to $\mathbb{V} := \{v \mid |v| \leq 0.475\}$, which corresponds to a choice of $\beta_2 := 0.95$. Figure 2 is a time plot of the states of the system for a choice of horizon length $N := 6$. As can be seen, the trajectories of the actual system with disturbances (5.1) remain in a 'tube' around the trajectory of the nominal system (5.2).

7. CONCLUSION

We have proposed a method for robust model predictive control of nonlinear systems with an additive bounded disturbance. The controller solves, at each time, two optimal control problems, one which solves a standard problem for the nominal system with tightened constraints (the solution of which defines a central path) and an ancillary problem (the solution of which steers the state of the towards the nominal trajectory thereby keeping the actual trajectories in a tube whose centre is the nominal trajectory). To keep the controller as simple as possible, we have chosen simple options for these two problems; for the nominal problem, we do not use the initial state as a decision variable, and for the ancillary problem we fix the terminal state rather than specifying a terminal cost function and terminal constraint set and discard the state constraints. Incorporation of a terminal cost function and terminal constraint set would increase complexity though other advantages might accrue.

Appendix A. OUTLINE OF PROOFS

Proposition 1: Inequality (4.5) follows from the postive definiteness of Q, (4.6) by assumption and (4.7) by standard arguments.

Proposition 2: (i) The proof of this result uses the fact that $\mathbf{v}^*(z)$ is also the global solution of $\mathbb{P}_N(z,z)$ ($\mathbf{u}^0(z,z) = \mathbf{v}^*(z)$) and that $V_N^0(z,z) = 0$. It then uses the implicit function theorem (Ortega and Rheinboldt, 1970) and the fact $\nabla_{\mathbf{u}} V_N(z,z,\mathbf{v}^*(z)) = 0$ and $\nabla_{\mathbf{uu}}(z,z,\mathbf{v}^*(z)) = \mathbf{R} > 0$ to show that there exists a neighbourhood $\{z\} \oplus \varepsilon(z)\mathcal{B}$ of z and a function $\nu_z(\cdot)$ such that $\nu_z(x)$ satisfies necessary conditions of optimality for $\mathbb{P}_N(x,z)$ at all x in this neighbourhood. It is then shown that $\varepsilon(z)$ can be chosen so that $\nu_z(x)$ is the global minimizer for $\mathbb{P}_N(x,z)$ in $\{z\}\oplus\varepsilon(z)\mathcal{B}$ and that $\nu_z(\cdot)$ is continuously differentiable. (ii) This result follows from similar arguments using the additional fact, easily established, that $\theta(\cdot)$ is continuously differentiable at $z = 0$.

Proposition 3: (i) This follows easily from (4.5). (ii) Using the Lipschitz continuity of $x \mapsto V_N^0(x,z)$ and (4.7) we obtain $V_N^0(f(x,\kappa_N(x,z) + w, f(x,\bar{\kappa}_N(z)) \le \gamma V_N^0(x,z) + k(z)|w| \le d$ if $|w| \le (1-\gamma)d/k(z)$ and $x \in S_d(z)$.

Proposition 4: This is a consequence of Proposition 3 (ii) and Definition 1.

Theorem 1: (i) Follows directly from Proposition 4. (ii) The infinite sequence $\{z(i)\}$ converges to the origin. Also, by Proposition 3(ii), every accumulation point $\hat{x}$ of the infinite sequence $\{x(i)\}$ satisfies $V_N^0(\hat{x},0) \le d$. Hence $\hat{x} \in S_d(0)$.

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