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Model Predictive Control based on Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Approach

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***Abstract*— A novel approach to the design of Model Predictive Control is proposed, using mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design method for time invariant discrete-time linear systems. The controller has the form of state feedback, satisfies quadratic input and state constraints and is constructed from the solution of a set of feasibility linear matrix inequalities. The control law takes account of disturbances naturally. A numerical example demonstrates the applicability of the algorithm.**

Keywords: constrained system, linear matrix inequalities (LMIs), model predictive control, $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm, disturbance rejection.

I. INTRODUCTION

Model predictive control (MPC) is a form of control in which the current control action is obtained by solving online, at each sampling instant, a finite horizon open-loop optimal control problem. Using the current state of the plant as the initial state; the optimization yields an optimal control sequence and the first control in this sequence is applied to the plant. This is its main difference from conventional control which uses a pre-computed control law. The advantages of MPC include ability to handle constraints, capability for controlling multivariable plants, just to name a few. MPC has received a lot of attention, because it presents good performances in such aspects as simplicity of computation mechanism and tracking properties [1]–[3]. It has also been used widely in practical applications to industrial process systems [4] and reconfigurable hardware such as FPGA chip [5]. In particular, it presents a proper control strategy for time invariant or time-varying systems or input/state constrained systems [6]–[8].

In most literature on MPC, the linear quadratic (LQ) optimization approach has been adopted. In recent years, there have been few attempts to construct a model predictive $\mathcal{H}_\infty$ controller for time-varying continuous/discrete linear systems, in which a dynamic game approach of minimizing worst case performance is adopted (in [9] terminal state constraint was used, in [10] quadratic terminal state weight was utilized, while in [11]–[13], matrix inequality conditions on the terminal weighting matrices for linear discrete/continuous varying systems were derived and a finite-horizon cost function was considered when disturbance was included). In a situation where the control signal acts against the worst possible disturbances, there is a close link between the $\mathcal{H}_\infty$ minimization and dynamic game approaches [14]. MPC has been applied to $\mathcal{H}_\infty$ problems in order to combine the practical advantage of MPC with the robustness of the $\mathcal{H}_\infty$ control, since robustness of MPC is still being investigated for it to be applied practically.

In this paper, we extend the result of [15] to constrained linear discrete-time invariant systems using a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design approach. This is more suitable as both performance and robustness issues are handled within a unified framework. The method presented in this paper develops an LMI design procedure for the state feedback gain matrix F, allowing input and state constraints to be included in a non-conservative manner. A main contribution, is the accomplishment of a prescribed disturbance attenuation in a systematic way by incorporating the well-known robustness guarantees through $\mathcal{H}_\infty$ constraints into MPC scheme. However, the issue of uncertainty in the model will be addressed in a future work.

This paper is organized as follows. In Section II we describe the system and give a statement of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. In Section III we derive sufficient conditions, in the form of LMIs, for the existence of a state feedback control law that achieves the design specifications. In section IV we consider a numerical example that illustrates our algorithm. Finally, we conclude in Section V.

II. PROBLEM FORMULATION

We consider the following discrete-time linear time invariant system:

$$\begin{aligned} x_{k+1} &= Ax_k + B_w w_k + B_u u_k \\ z_k &= \begin{bmatrix} C_z x_k \\ D_{zu} u_k \end{bmatrix} \\ x_0 &\ \ given, \end{aligned} \tag{1}$$

where $x_k \in \mathcal{R}^n$ is the state, $w_k \in \mathcal{R}^{n_w}$ the disturbance, $u_k \in \mathcal{R}^{n_u}$ the control, $z_k \in \mathcal{R}^{n_z}$ the controlled output, and $A \in \mathcal{R}^{n\times n}$, $B_w \in \mathcal{R}^{n\times n_w}$, $B_u \in \mathcal{R}^{n\times n_u}$, $C_z \in \mathcal{R}^{n_{z_1}\times n}$ and $D_{zu} \in \mathcal{R}^{n_{z_2}\times n_u}$ and where $n_z = n_{z_1} + n_{z_2}$.

We assume that the pair (A, B_u) is stabilizable and that the disturbance is bounded as

$$\|w\|_2 := \sqrt{\sum_{k=0}^{\infty} w_k^T w_k} \le \bar{w} \tag{2}$$

where $\bar{w} > 0$ is known.

The aim is to find a state feedback control law $\{u_k = Fx_k\}$ in $\mathcal{L}_2$, where $F \in \mathcal{R}^{n_u\times n}$, such that the following constraints are satisfied:

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1) The transfer matrix from w to z, denoted as T_{zw} is stable and for given $\gamma > 0$ the $\mathcal{H}_\infty$ constraint
$$\|T_{zw}\|_\infty < \gamma \quad (3)$$
is satisfied.

2) For given $\alpha > 0$ the $\mathcal{H}_2$ constraint
$$\|z\|_2 := \sqrt{\sum_{k=0}^{\infty} z_k^T z_k} < \alpha, \quad (4)$$
is satisfied.

3) For given $H_1,\ldots,H_{m_u} \in \mathcal{R}^{n_h \times n_u}$ and $\bar{u}_1,\ldots,\bar{u}_{m_u} > 0$ the input constraints
$$u_k^T H_j^T H_j u_k \leq \bar{u}_j^2, \quad \forall k; \text{ for } j = 1,..,m_u, \quad (5)$$
are satisfied.

4) For given $E_1,...,E_{m_x} \in \mathcal{R}^{n_e \times n}$ and $\bar{x}_1,\ldots,\bar{x}_{m_x} > 0$ the state/output constraints
$$x_{k+1}^T E_j^T E_j x_{k+1} \leq \bar{x}_j^2, \ \forall k; \text{ for } j=1,\ldots,m_x, \quad (6)$$
are satisfied.

An $F \in \mathcal{R}^{n_u \times n}$ satisfying these requirements will be called an admissible state feedback gain.

III. LMI FORMULATION OF SUFFICIENCY CONDITIONS

The next theorem, which is the main result of this paper, derives sufficient conditions, in the form of LMIs, for the existence of an admissible F.

Theorem 1: Let all variables, definitions and assumptions be as above. Then there exists an admissible state feedback gain matrix F if there exists solutions $Q = Q^T \in \mathcal{R}^{n \times n}$ and $Y \in \mathcal{R}^{n_u \times n}$ to the following LMIs
$$\begin{bmatrix} -Q & \star & \star & \star & \star \\ 0 & -\alpha^2\gamma^2 I & \star & \star & \star \\ AQ+B_uY & \alpha^2 B_w & -Q & \star & \star \\ C_zQ & 0 & 0 & -\alpha^2 I & \star \\ D_{zu}Y & 0 & 0 & 0 & -\alpha^2 I \end{bmatrix} < 0 \quad (7)$$
$$\begin{bmatrix} 1 & \star & \star \\ \gamma^2\bar{w}^2 & \alpha^2\gamma^2\bar{w}^2 & \star \\ x_0 & 0 & Q \end{bmatrix} \geq 0 \quad (8)$$
$$\begin{bmatrix} \bar{u}_j^2 I & \star \\ Y^T H_j^T & Q \end{bmatrix} \geq 0, j=1,\ldots,m_u \quad (9)$$
$$\begin{bmatrix} \bar{x}_j^2 I-(1+\bar{w}^2)E_jB_wB_w^TE_j^T & \star \\ QA^TE_j^T+Y^TB_u^TE_j^T & \frac{Q}{(1+\bar{w}^2)} \end{bmatrix} \geq 0, j=1,\ldots,m_x \quad (10)$$
where $\star$ represents terms readily inferred from symmetry. If such solutions exist, then
$$F = YQ^{-1}.$$

Proof: Using $u_k = Fx_k$, the dynamics in (1) become
$$x_{k+1} = \overbrace{(A + B_uF)}^{A_{cl}} x_k + B_w w_k, \quad z_k = \overbrace{\begin{bmatrix} C_z \\ D_{zu}F \end{bmatrix}}^{C_{cl}} x_k \quad (11)$$

Consider a quadratic function $V(x) = x^TPx$, $P > 0$ of the state x_k. It follows from (11) that
$$\begin{aligned} V(x_{k+1}) - V(x_k) &= x_k^T[A_{cl}^TPA_{cl} - P]x_k \\ &\quad + x_k^TA_{cl}^TPB_ww_k + w_k^TB_w^TPA_{cl}x_k \\ &\quad + w_k^TB_w^TPB_ww_k \\ &= \begin{bmatrix} x_k^T & w_k^T \end{bmatrix} K \begin{bmatrix} x_k \\ w_k \end{bmatrix} \\ &\quad - x_k^TC_{cl}^TC_{cl}x_k + \gamma^2 w_k^T w_k, \end{aligned} \quad (12)$$
where
$$K = \begin{bmatrix} A_{cl}^TPA_{cl} - P + C_{cl}^TC_{cl} & A_{cl}^TPB_w \\ B_w^TPA_{cl} & B_w^TPB_w - \gamma^2 I \end{bmatrix}. \quad (13)$$
Assuming that $\lim_{k\to\infty} x_k = 0$ we have
$$\sum_{k=0}^{\infty}[x_{k+1}^TPx_{k+1} - x_k^TPx_k] = -x_0^TPx_0. \quad (14)$$
We write the $\mathcal{H}_2$ cost function as
$$\|z\|_2^2 = \sum_{k=0}^{\infty}(x_k^TC_{cl}^TC_{cl}x_k - \gamma^2 w_k^Tw_k) + \gamma^2\sum_{k=0}^{\infty} w_k^Tw_k. \quad (15)$$
Adding (14) and (15) and carrying out a simple manipulation gives
$$\|z\|_2^2 = x_0^TPx_0 + \gamma^2\|w\|_2^2 + \sum_{k=0}^{\infty}\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} K \begin{bmatrix} x_k \\ w_k \end{bmatrix} \quad (16)$$
where K is defined in (13).

An application of the bounded real lemma [16] shows that A_{cl} is stable and (3) is satisfied if and only if there exists $P = P^T > 0$ such that
$$K < 0. \quad (17)$$
Next, we linearize the matrix inequality $K < 0$. This can be written as
$$\begin{bmatrix} -P & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} A_{cl}^T \\ B_w^T \end{bmatrix} P \begin{bmatrix} A_{cl} & B_w \end{bmatrix} + \begin{bmatrix} C_z^T \\ 0 \end{bmatrix}\begin{bmatrix} C_z & 0 \end{bmatrix} + \begin{bmatrix} F^TD_{zu}^T \\ 0 \end{bmatrix}\begin{bmatrix} D_{zu}F & 0 \end{bmatrix} < 0.$$
Using Schur complement, this is equivalent to
$$\begin{bmatrix} -P & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ A_{cl} & B_w & -P^{-1} & \star & \star \\ C_z & 0 & 0 & -I & \star \\ D_{zu}F & 0 & 0 & 0 & -I \end{bmatrix} < 0.$$
Pre- and post-multiplying by diag (P^{-1}, I, I, I, I),
$$\begin{bmatrix} -P^{-1} & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ A_{cl}P^{-1} & B_w & -P^{-1} & \star & \star \\ C_zP^{-1} & 0 & 0 & -I & \star \\ D_{zu}FP^{-1} & 0 & 0 & 0 & -I \end{bmatrix} < 0.$$
Setting $Q = \alpha^2P^{-1}$, $F = YP\alpha^{-2} = YQ^{-1}$ and multiplying through by α^2, we get (7).

Now, it follows from (2), (16) and (17) that

$$\|z\|_2^2 \le x_0^T P x_0 + \gamma^2 \|w\|_2^2 \le x_0^T P x_0 + \gamma^2 \bar{w}^2.$$

Thus the $\mathcal{H}_2$ constraint in (4) is satisfied if

$$x_0^T P x_0 + \gamma^2 \bar{w}^2 < \alpha^2.$$

Dividing by α^2, rearranging and using a Schur complement gives (8) as an LMI sufficient condition for (4).

To turn (5) and (6) into LMIs we first show that $x_k^T P x_k \le \alpha^2 \ \forall k > 0$. Since $K < 0$, it follows from (12) that

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k \le \gamma^2 w_k^T w_k.$$

Applying this inequality recursively, we get

$$\begin{aligned} x_k^T P x_k &\le x_0^T P x_0 + \gamma^2 \sum_{j=0}^{k-1} w_j^T w_j \\ &\le x_0^T P x_0 + \gamma^2 \bar{w}^2 \le \alpha^2. \end{aligned}$$

It follows that

$$\|P^{\frac{1}{2}} x_k\|^2 \le \alpha^2, \tag{18}$$

or equivalently,

$$x_k^T Q^{-1} x_k \le 1, \ \forall k > 0. \tag{19}$$

Next, we transform the constraints in (5) to a set of LMIs as follows: Setting $F = YQ^{-1} = YP\alpha^{-2}$ and $u_k = Fx_k$,

$$\begin{aligned} u_k^T H_j^T H_j u_k &= x_k^T F^T H_j^T H_j F x_k \\ &= \alpha^{-4} x_k^T P Y^T H_j^T H_j Y P x_k \\ &= \alpha^{-4} x_k^T P^{\frac{1}{2}} P^{\frac{1}{2}} Y^T H_j^T H_j Y P^{\frac{1}{2}} P^{\frac{1}{2}} x_k, \end{aligned}$$

and using (18),

$$\begin{aligned} u_k^T H_j^T H_j u_k &\le \alpha^{-4} \|P^{\frac{1}{2}} x_k\|^2 \|P^{\frac{1}{2}} Y^T H_j^T H_j Y P^{\frac{1}{2}}\| \\ &\le \alpha^{-2} \|P^{\frac{1}{2}} Y^T H_j^T H_j Y P^{\frac{1}{2}}\| \\ &= \alpha^{-2} \lambda_{max}(P^{\frac{1}{2}} Y^T H_j^T H_j Y P^{\frac{1}{2}}) \\ &= \alpha^{-2} \lambda_{max}(H_j Y P Y^T H_j^T). \end{aligned}$$

where $\lambda_{max}(\cdot)$ denotes the largest eigenvalue. It follows that a sufficient condition for (5) is

$$\alpha^{-2} \lambda_{max}(H_j Y P Y^T H_j^T) \le \bar{u}_j^2.$$

Using a Schur complement, this is equivalent to the LMI in (9). Finally, to obtain an LMI formulation of the state constraints (6), the following steps are carried out:

$$\begin{aligned} & x_{k+1}^T E_j^T E_j x_{k+1} \\ &\quad = (A_{cl} x_k + B_w w_k)^T E_j^T E_j (A_{cl} x_k + B_w w_k) \\ &\quad = \begin{bmatrix} Q^{-\frac{1}{2}} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} Q^{\frac{1}{2}} A_{cl}^T \\ B_w^T \end{bmatrix} E_j^T E_j \begin{bmatrix} A_{cl} Q^{\frac{1}{2}} & B_w \end{bmatrix} \begin{bmatrix} Q^{-\frac{1}{2}} x_k \\ w_k \end{bmatrix} \\ &\quad \le (1+\bar{w}^2) \lambda_{max} \left(E_j \begin{bmatrix} Q A_{cl}^T \\ B_w^T \end{bmatrix}^T \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q A_{cl}^T \\ B_w^T \end{bmatrix} E_j^T \right), \end{aligned}$$

since (19) and $\|w\|_2 \le \bar{w}$ imply that

$$\left\| \begin{bmatrix} Q^{-\frac{1}{2}} x_k \\ w_k \end{bmatrix} \right\|^2 \le (1+\bar{w}^2).$$

It follows that a sufficient condition for (6) is

$$\bar{x}_j^2 I - (1+\bar{w}^2) \left(E_j \begin{bmatrix} A_{cl} Q & B_w \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q A_{cl}^T \\ B_w^T \end{bmatrix} E_j^T \right) \ge 0.$$

Using a Schur complement, the above inequality is equivalent to the LMI in (10). This completes the proof. ■

Remark 1: In the absence of disturbances our result reduces to that in [15]. This corresponds to setting $B_w = 0$.

Remark 2: Note that, for fixed γ^2 (or α^2), minimizing α^2 (or γ^2) is an LMI optimization problem.

Remark 3: Note that this is not a standard $\mathcal{H}_\infty$ problem; however, a stabilizing controller is called γ-suboptimal [17] if the obtained closed-loop system fulfils the γ-disturbance attenuation.

IV. Numerical Example

In this section, we present an example to illustrate the effectiveness of the proposed model predictive mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control algorithm. We consider a scalar systems as follows:

$$x_{k+1} = x_k + u_k + w_k, \quad x_0 = 1,$$

so that $A = 1$, $B_u = 1$ and $B_w = 1$ and initial state $x_0 = 1$. The input constraint is $|u_k| \le 1$. For the cost function we have set $D_{zu} = 1$ and $C_z = 1$. For the disturbance, we considered a persistent disturbance of the form $w_i = 0.1$ for all $0 \le i \le 10$.

Figures 1 and 2 and 3 compare the closed-loop response of our algorithm with that of [15]. In Figure 1, we set a low disturbance rejection level by setting $\gamma^2 = 20$. Note that the responses for both algorithm are very close. In Figure 2, we set $\gamma^2 = 2.12$, which is the lowest value for which a feasible solution exists. Note that the performance and response of the system based on the new method was better than that obtained using the method in [15], since it has a smaller settling time and smaller steady-state error. Constraints on the input were satisfied by the two methods; however the approach of [15] was more conservative with respect to the control signal.

In addition we confirmed Remark 1, by setting the value of B_w to zero, so that there are no disturbances. The responses were exactly the same, with conservativeness in control, as well as longer settling time. This is depicted in Figure 3.

V. Conclusion

In this paper, we proposed a model predictive mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design technique for time invariant discrete-time linear systems subject to constraints on the inputs and/or states. This method takes account of disturbances naturally by imposing the $\mathcal{H}_\infty$-norm constraint in (3) and thus extends the work in [15]. The development is based on full state feedback assumption and the on-line optimization involves the solution of an LMI-based linear objective minimization.

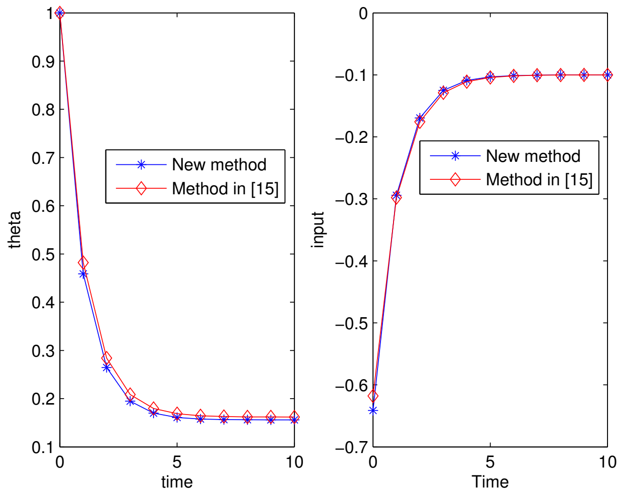

Fig. 1. Closed-loop responses, $\gamma^2 = 20$

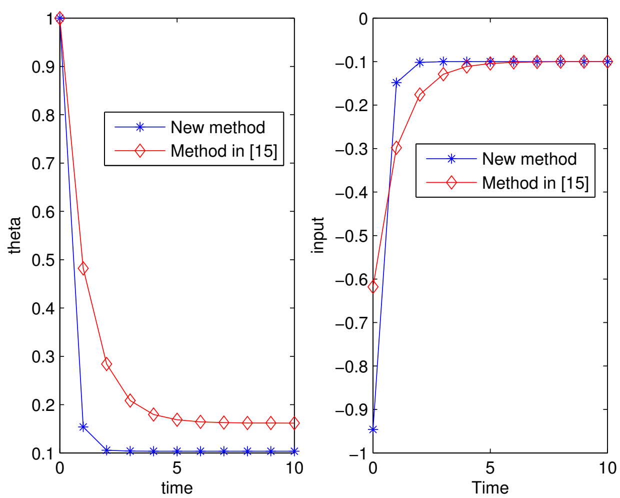

Fig. 2. Closed-loop responses , $\gamma^2 = 2.12$

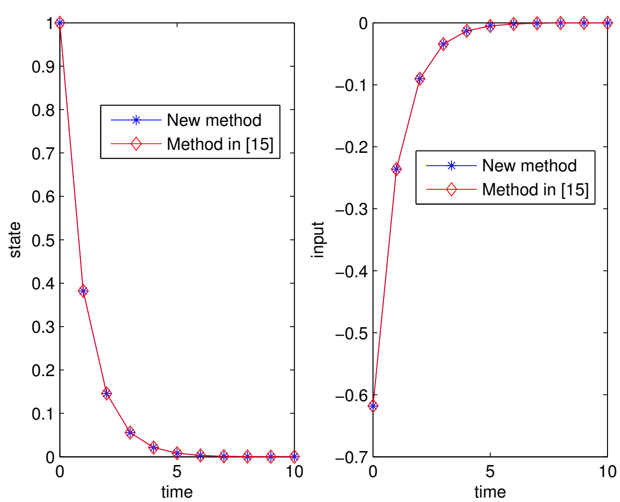

Fig. 3. Closed-loop responses for system with $B_w = 0$

The resulting time-invariant state-feedback control law minimizes an upper bound on the objective performance at each time step. The new approach reduces to that of [15] when there are no disturbances present in the system.

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